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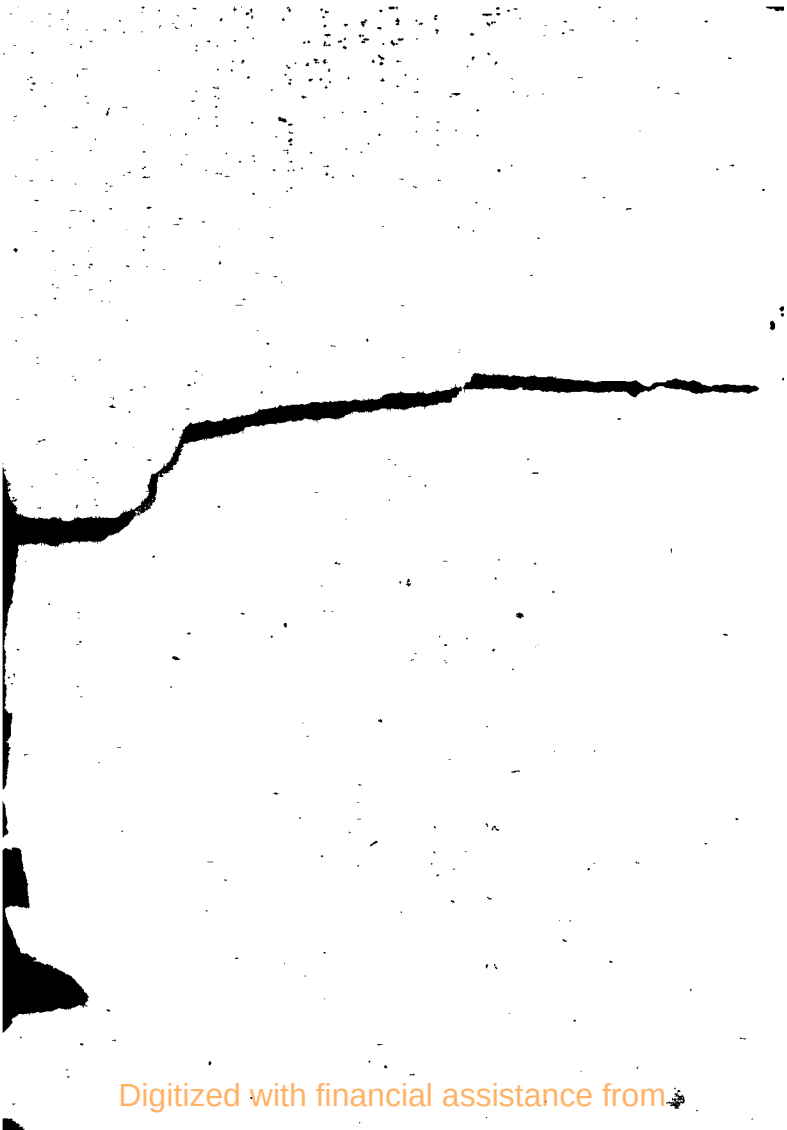

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A

TREATISE

ON

ISOPERIMETRICAL PROBLEMS,

AND THE

CALCULUS OF VARIATIONS.

19824

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PREFACE.

THERE needs, perhaps, no other apology for the present Work, than the mere statement of the fact, that there is, on the same subject, no English, and only one Foreign Treatise *, of which the celebrated Euler is the Author.

The copies of this last, a work of uncommon merit, are in this country very rare. But, notwithstanding its various excellencies, a mere translation of it would not have rendered unnecessary the present treatise, since, independently of any other consideration, several of Euler's most important processes are, by more concise ones, now superseded.

But although there is only one separate Treatise, the subject itself has been discussed in several analytical works, English as well as

* Methodus inveniendi Lineas curvas proprietate maximi minimive gaudentes, Lausanne, 1744.

Foreign. Our countrymen are accustomed to assign, in their treatises on fluxions, a chapter to the maxima and minima of curves; and, the foreign mathematicians consider the same subject under the head of "*Calcul des Variations.*"

Maclaurin, Simpson, Emerson, especially the two latter, have not explained the principles of the subject with sufficient perspicuity and precision; and, in point of depth of research, there are various problems, to the solution of which, their formulæ are totally inadequate. Indeed, which is remarkable, there is, I believe, no English treatise that mentions Euler's formula of solution; which, for the simplicity of its construction, and facility of application, is exceeded by none in the whole compass of analytical science.

The foreign mathematicians, Euler, Le Seur, Bossut, and Lacroix, have furnished their treatises on the differential and integral calculi, with formulæ of solution adequate to any case that can be proposed. There is no deficiency on that head. But, these Authors too suddenly carry the reader into the middle of the business and immerse him in calculations; and, if they soon provide him with the instruments of solution,

they instruct him neither in the object nor the principle of their construction.

This is not said in the spirit of an Author preparing, by the censure of preceding Works, a reception for his own. For the truth of the statement, I would appeal to the experience of all who have consulted the above cited Authors; besides, it is not difficult to assign the reason of the alledged defects.

When Lagrange in 1760, published his new method of solving problems of maxima and minima, he composed his memoir for mathematicians, familiar with its subject, and well versed in the researches of the Bernoullis and of Euler. Accordingly, he very briefly states the principles of his calculus, and enters into no explanation of the nature of the subject. His compendious method of computation, however, has been adopted; and subsequent Authors have composed their treatises very much on the plan of Lagrange's memoir, with some, but slight and imperfect, preliminary explanation. These Treatises, however, the student is expected to understand; that is, if the matter be fairly stated, he is expected to understand an intricate subject, with advantages much less than consum-

mate mathematicians before him enjoyed; since, there is neither proper explanations presented to him, nor is he directed, by way of preparation, previously to consult the Works of Euler and the Bernoullis.

Such are, in my opinion, the defects of existing methods; still, however, I have not composed a treatise on the subject, by merely remedying them; that is, by inserting formulæ of sufficient extent, and by more fully explaining and illustrating their principles. But, on a novel plan, I have combined the historical progress with the scientific developement of the subject; and endeavoured to lay down and inculcate the principles of the Calculus, whilst I traced its gradual and successive improvements.

If this has been effected, which I think it has, in a compass not very wide of that which a strictly scientific treatise would have required, the only serious objection against the present plan is, in part, obviated. For, there is little doubt, the student's curiosity and attention will be more excited and sustained, when he finds history blended with science, and the demonstration of formulæ accompanied with the object and the causes of their invention, than by a mere analytical exposition of the principles of the subject.

The plan, perhaps, would not suit any other department of science, so well as it does this; which is limited in its extent, and has had but few, although eminent cultivators.

Other advantages, besides that of an excited attention, may accrue to the student from the present plan. He will have an opportunity of observing how a calculus, from simple beginnings, by easy steps, and seemingly the slightest improvements, is advanced to perfection; his curiosity too, may be stimulated to an examination of the works of the contemporaries of Newton; works once read and celebrated: yet the writings of the Bernoullis are not antiquated from loss of beauty, nor deserve neglect, either for obscurity, or clumsiness of calculation, or shallowness of research. Their processes indeed are occasionally somewhat long, and want the trim form of modern solution. They are not, however, therefore the less adapted to the student, who is solicitous for just and full views of science, rather than for neat novelties and mere store of results. Indeed, the Authors who write near the beginnings of science, are, in general the most instructive: they take the reader more along with them, shew him the real difficulties, and, which is a main point, teach

him the subject, the way by which they themselves learned it.

In a former Work*, I adopted the foreign notation, and the present occasion furnishes some proof of the propriety of that adoption. In the Calculus of Variations, it is necessary to have symbols denoting operations, similar to those that take place in the Differential Calculus: now, d being the symbol for the latter, δ is a most convenient one for the former: analogous to δ there is no symbol in the English system of notation. If then I had used the fluxionary notation with points or dots, I must have invented symbols corresponding to δ and the characters formed by means of it. But, the invention of merely new symbols is in itself an evil. M. Lagrange indeed, whose power over symbols is so unbounded that the possession of it seems to have made him capricious, has treated the subject of Variations without the foreign notation; this he rejects altogether; and, which is strange, has employed the English notation, but not adopted its signification. Thus, with him, $\dot{x}$ is not the fluxion, but the variation of x: the fluxions or differentials of quantities are not

* Principles of Analytical Calculation, 1803.

expressed by him, but solely the fluxionary or differential coefficients; thus, if u be a function of x, u' $\left(=\frac{du}{dx} \text{ or } = \frac{\dot{u}}{\dot{x}}\right)$ is the differential coefficient. What advantages are to arise from these alterations it is not easy to perceive: yet they ought to be great, to balance the plain and palpable evils of a confusion in the signification of symbols, and of the invention of a system of notation to represent what already was represented with sufficient precision. No authority can even sanction so capricious an innovation.

There is another point towards which I am not unwilling to draw the attention of the reader; and that is, the method of demonstration by geometrical figures. In the first solution of Isoperimetrical problems, the Bernoullis use diagrams and their properties. Euler, in his early essays, does the same; then, as he improves the calculus he gets rid of constructions. In his Treatise *, he introduces geometrical figures, but almost entirely, for the purpose of illustration: and finally, in the tenth volume of the *Novi Comm. Petrop.* as Lagrange had done in the *Miscellanea Taurinensea*, he

* Methodus inveniendi, &c.

expounds the calculus, in its most refined state, entirely without the aid of diagrams and their properties. A similar history will belong to every other method of calculation, that has been advanced to any degree of perfection.

The plan of the Work has been slightly described; and indeed it scarcely requires any explanation. On that, however, I chiefly rely, as the means of rendering easy a subject, which is acknowledged to be one of the most difficult. But, although I am not aware of having omitted any thing that is requisite to the full explanation of the subject, yet I cannot flatter myself that it will be thoroughly understood from this Work alone. For, in general it may be laid down as true, that no doctrine, of novelty and intricacy, can be completely taught by a single Treatise. It seems to be indispensably necessary for the student, that the subject should be put under several points of view: that if not apprehended under one, it may be under another. For this reason, though not wanting an Author's partiality for his own performance, I recommend the perusal of those works, to which frequent reference is made in the following pages.

I return my sincere thanks to the Syndics of the University Press, who, very liberally, have defrayed two-thirds of the expense of the present publication.

Caius College,
Nov. 17, 1810.

TABLE *of the* FOREIGN, *and the corresponding* ENGLISH *Notation.*

F.	dx	d^2x	d^3x	d^nx	dx^2	dx^3
E.	$\dot{x}$	$\ddot{x}$	$\dddot{x}$	$\overset{n}{x}$	$\dot{x}^2$	$\dot{x}^3$
F.	$\frac{du}{dx}$	$\frac{d^2u}{dx^2}$	$\frac{d^3u}{dx^3}$	$\frac{d^2u}{dx.dy}$	$\frac{d^3u}{dx^2.dy}$	$d^3\left(\frac{dy}{dz}\right)$
E.	$\frac{\dot{u}}{\dot{x}}$	$\frac{\ddot{u}}{\dot{x}^2}$	$\frac{\dddot{u}}{\dot{x}^3}$	$\frac{\ddot{u}}{\dot{x}\dot{y}}$	$\frac{\dddot{u}}{\dot{x}^2\dot{y}}$	$\left(\frac{\dot{y}}{\dot{z}}\right)^{\cdot\cdot}$

List of Foreign Authors that treat of the Subject of the present Work.

BERNOULLI, James and John. — Works.

EULER.
- Comm. Acad. Petrop. tom. VI. VIII. 1733, 1736.
- Methodus Inveniendi Lineas curvas proprietate maximi minimive gaudentes, *Lausanne. Geneva,* 1744.
- Novi Comm. Petrop. tom. X. 1766.
- Calcul Integral, 1768.

LAGRANGE.
- Miscellanea Taurinensia, tom. II. IV.
- Théorie des Fonctions Analytiques, 1797.
- Leçons sur le Calcul des Fonctions, 1806.

BORDA, et FONTAINE.—Acad. des Sciences, *Paris,* 1767.
LE SEUR et JACQUIER.—Elemens de Calcul Integral, 1768.
SAURI.—Cours de Mathematiques, tom. V. 1774.
LEGENDRE.—Acad. des Sciences, *Paris,* 1786.
LACROIX.—Calc. Diff. et Integral, 1797.
BOSSUT.—Calc. Diff. et Integral, 1798.

ON

ISOPERIMETRICAL PROBLEMS,

AND THE

CALCULUS *of* VARIATIONS.

CHAP. I.

The Problem of the Curve of Quickest Descent proposed.—James Bernoulli's Solution of it.—Principle of that Solution.

THE ordinary questions of maxima and minima were amongst the first that engaged the attention of mathematicians at the time of the invention of the Differential or Fluxionary Calculus. This invention took place about the year 1684, three years before the memorable æra of the publication of the Principia. But, the principles of the differential-calculus, were not, like those of physical astronomy, given to the world, at once, and as it were, on a sudden, in a formal treatise. They were communicated gradually and by piecemeal; in letters between men of science, in small essays and tractates, in solutions of particular problems published chiefly in

a work intitled the *Acta Eruditorum* edited at Leipsick*; which, at that time, was the common vehicle of communication between men of science and literature. There is scarcely a formula or process, which we now find compactly stated in our modern treatises, that was not in those Acts the subject of some correspondence and discussion. The methods of drawing tangents, radii of curvature, of determining the points of inflexion, the maxima and minima of quantities, &c. were treated as subjects of great importance by Leibnitz, and by the two Bernoullis.

The first problem relative to a species of maxima and minima distinct from the ordinary, was proposed by Newton in the Principia: it was, that of the solid of least resistance. But, the subject and doctrine became not matter of discussion and controversy, till John Bernoulli, Professor of mathematics at Groningen, required of mathematicians the determination of the curve of quickest descent. This he did in the *Leipsick Acts* for June, 1696†, p. 269, under the following form:

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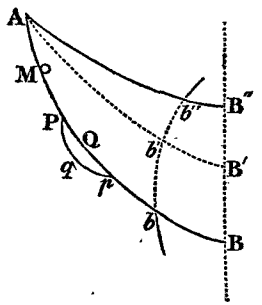
* Problems proposed and solved, became an amazing means of promoting the methods of calculation, or the calculus, to which they belonged. Of this John Bernoulli was aware, when, in his famous Programma relative to the line of quickest descent, he says, "Cum compertum habeamus, vix quicquam esse quod magis excitet generosa ingenia, ad moliendum quod conducit augendis scientiis, quam difficilium pariter, et utilium quæstionum propositionem." *Opera*, tom. I. p. 166.

† Opera, tom. I. p. 155.

"PROBLEMA NOVUM,

ad cujus Solutionem Mathematici invitantur.

"Datis in plano verticali duobus punctis *A* et *B*,
"assignare mobili *M* viam *AMB*, per quam gravitate



"sua descendens, et moveri incipiens a puncto *A*, bre-
"vissimo tempore perveniat ad alterum punctum *B*."

Six months, the time allotted for its solution, being elapsed, and no solution appearing, John Bernoulli at the request of Leibnitz (who intimated that he had solved the problem) prorogued the term, and again anounced the problem in a *programma** edited at Groningen in January 1697. In the May following, his brother, James Bernoulli, professor of mathematics at Basle, published its solution in the *Acta Erud. Lips.* p. 211†; and after the following manner:

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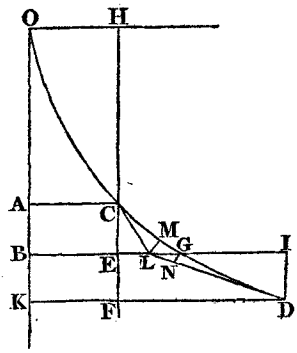
* Opera, tom. I. p. 166.

† See also his Works, tom. II. p. 768.

Let OGD be the curve; conceive a portion of it CGD, to be divided into two parts CG, GD; and take another



element to the curve CLD, divided also into two parts CL, LD, and indefinitely near to CGD: then since by hypothesis, the time through $CG+GD$, is to be a minimum, and since quantities at or near their state of minimum may be considered constant (for their increments or decrements are very small), we have

$$t.CG+t.GD=t.CL+t.LD$$

[$t.CG$, abridgedly representing the time through CG.]

and $\therefore$ $t.CG-t.CL=t.LD-t.GD$

again $CE : CG :: t.CE : t.CG$

[CG considered as an inclined plane.]

and $CE : CL :: t.CE : t.CL$

consequently, $CE : CG-CL\ [MG] :: t.CE : t.CG-t.CL$

but $MG : LG :: EG : CG$

[by similar Δs LMG, GCE.]

$\therefore$ $CE : LG :: EG \times (t.CE) : CG \times (t.CG-t.CL)$

similarly $EF : LG :: GI \times (t.EF) : GD \times (t.LD-t.GD.)$

Hence, equating the two values of LG, we have

$$EG \times t.CE \times EF \times GD = GI \times t.EF \times CE \times CG$$

$$\text{or } EG \times \frac{CG}{\sqrt{HC}} \times EF \times GD = GI \times \frac{GD}{\sqrt{HE}} \times CE \times CG$$

[substituting for $t.CE$, $t.EF$.]

$$\text{or } \frac{EG}{\sqrt{HC}} : \frac{GI}{\sqrt{HE}} :: CE : EF :: CG : GD \quad [1.]$$

which is a property of the cycloid *: hence the curve of quickest descent, or the *brachystochrone*, is a cycloid.

The proportion [1] may be thus expressed:

$$\frac{EG}{GC} : \frac{GI}{GD} :: \sqrt{HC} : \sqrt{HE}, \quad \text{or}$$

sin. $\angle ECG$: sin. $\angle GDI$ :: $\sqrt{HC}$: $\sqrt{HE}$ :: vel^{y}. at C : vel^{y}. at G, or the sine of the angle made by a vertical and an element of the curve is proportional to the velocity.†

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* This will easily appear by constructing the figure with its generating circle; or, the known equation of the cycloid may be thus deduced: since $\frac{EG}{\sqrt{HC}.CG} = \frac{GI}{\sqrt{HE}.GD}$ we have $\frac{dx}{\sqrt{y}.dz} = \frac{dx'}{\sqrt{y'}.dz'}$ and since dx', y', dz', are the consecutive values of dx, y, dz, it follows from the above equation, that $\frac{dx}{\sqrt{y}.dz}$ is every where the same, or is constant, and therefore may be put $= \frac{1}{\sqrt{a}}$. Hence, $a\,dx^2 = y.dx^2 + y.dy^2$ and $dx = \frac{\sqrt{y}}{\sqrt{(a-y)}}dy$, the equation to a cycloid.

† Lagrange, *Cal. des Fonct.* 8vo. p. 425, says that this was the form of the result obtained by Bernoulli. It is not however the form given in the first solution that occurs in Bernoulli's Works.

In the preceding demonstration two principles are involved: one * borrowed from the doctrine of the ordinary maxima and minima of quantities: the other, at the time of the first discussion of these questions entirely new, and in fact assuming; That the property appertaining to the whole curve, belongs also to any element of the curve: for of the preceding problem, the condition was, that the time down the whole curve OCD should be a minimum, that is, less than down any other curve, passing through the points O, D: but in the demonstration, an element CGD is taken, and it is assumed that the time down this element is also a minimum: so that, the whole curve is determined, by determining a portion or element of it. The principle undoubtedly is true, both in the preceding and in other problems, but it is not universally true. Of the exceptions, and of their reasons, we shall hereafter speak, and here only briefly notice, that the Bernoullis and Brook Taylor, preface their small tracts on this subject, with this principle, establishing the same nearly after the following manner:

If the curve APB (see fig. p. 3.) has a certain property of minimum, a portion of it PQp, has the same property: for suppose AP, Bp, to be determined, then if PQp possesses not the property of maximum or minimum, suppose another small portion Pqp to possess it; therefore Pqp added to AP, and Bp, will form a curve, that in the case of maximum will contain more,

* See page 4. line 5, &c.

or of minimum, less than $APQpB$; which is contrary to the hypothesis *.

The first principle is, as it has been remarked, that which is employed in the ordinary questions of maxima and minima. If, for instance, we wish to determine when an ordinate y is a minimum, dy its differential is put equal to nothing. Hence, the contiguous or consecutive ordinate which generally is,

$$y \pm \frac{dy}{dx} . \Delta x + \frac{d^2y}{2.dx^2} (\Delta x)^2 \pm \&c. \dagger$$

is in this case reduced to

$$y + \frac{d^2y}{2.dx^2} . (\Delta x)^2 \pm \&c.$$

which, if we make Δx infinitely small, will differ from the ordinate y only by infinitely small quantities of the second order; and if we neglect them, may be said to be equal to the ordinate y. Of this nature, and requiring this explanation, is the equality in James Bernoulli's demonstration, when the time through CG + the time through GD = the time through CL + the time through LD.

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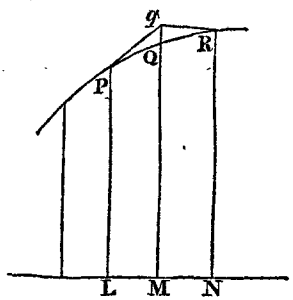
* P. 226. tom. II. John Bern. *Leip. Act.* 1701, p. 213. James Bernoulli, the author of the essay, as if suspicious of the truth of his principle, says, "Sensus theorematis vel demonstrationis ejus videtur paulo obscurior, nec satis determinatus: sed planior et infra ex applicatione, &c." John Bernoulli also, *Acad. des Sciences*, 1706, p. 236. lays down the same principle: "Parceque toute courbe qui doit donner un maximum, conserve aussi dans toutes ses parties les loix de ce même maximum, &c."

† Woodhouse, *Prin. Analy. Calc.* pp. 44, 163.

The use of the principle, however, in Bernoulli's problem, and its illustration, will enable us to point out the real and essential distinction between the ordinary calculus of maxima and minima, and that which is the subject of the present treatise. In the former the relation of y to x is supposed to be given, or, in other words, the function f in the equat^{n}. $y = fx$ is supposed to be known; and the process of equating y and $y \pm \frac{dy}{dx} . \Delta x +$ &c., or of making $dy = 0$, enables us to find a particular value of x which substituted in $y = fx$ gives the greatest or least value of y. Thus in the equation $y = \surd(2ax - x^2 + b)$, $fx = \surd(2ax - x^2 + b)$ and if the greatest ordinate



be required, (putting $MQ = y$, and $MN = \Delta x$), $NR = y + \frac{dy}{dx} . \Delta x . + \frac{d^2y}{2 . dx^2}(\Delta x^2) +$ &c. and $\frac{dy}{dx} = \frac{a - x}{y} = 0$, consequently $x = a$, and the maximum value of $y = \surd(a^2 + b)$. In the latter calculus, on the contrary, the relation of y to x is not given but sought, or in other words the form of the function fx [that is, whether it is $\surd(2ax - x^2)$ or e^{ax} or $(ax^n - bx^m)^{\frac{1}{r}}$, &c.] is the object of investigation.

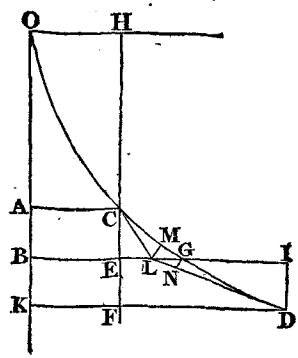
For this reason, the augmented value of $y = MQ$ is not, as before, its consecutive value $NR = y + \Delta y$, but $Mq = y + Qq = y + v$, which must be substituted for y in expanding any function of y. Thus, if the equation between y and x, or in other words, if the function fx is required, such that $\int (ax - y^2)\, y.dx$ shall be a maximum, we have, changing y into $y + v$, the consecutive value of the expression for the maximum, $(axy - y^3) + (ax - 3y^2)v +$ &c. and since this expression must be a maximum, the second term, analogous to $\frac{dy}{dx}\Delta x$ in the former instance, must $= 0$, or $ax - 3y^2 = 0$, and $y = \sqrt{\frac{ax}{3}}$.

Here it is plain, the result gives the *form* of the equation, that must subsist between y and x, so that generalizing, it may be said, that it is the form of the function fx in $y = fx$, which is the object of search. The two preceding instances plainly shew the points of agreement and disagreement of the two calculi.

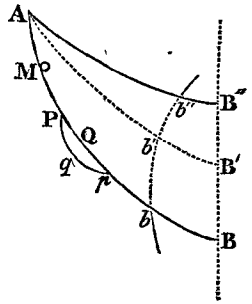
Before we quit this part of our subject we will shew how Bernoulli's method of solution, without any alteration of principle, may be abridged. Since the time varies as the space divided by the velocity, and the velocity as the square root of the height, we have,

$\frac{CL}{\sqrt{HC}} + \frac{LD}{\sqrt{HE}} = \frac{CG}{\sqrt{HC}} + \frac{GD}{\sqrt{HE}}$, from the principle of minimum, $\therefore \frac{LC - (CL + MG)}{\sqrt{HC}} = \frac{(LD - LN) - LD}{\sqrt{HE}}$,

or $\frac{MG}{\sqrt{HC}} = \frac{LN}{\sqrt{HE}}$, or from similar triangles, $\frac{EL}{CL\sqrt{HC}} \cdot LG = \frac{LI}{LD\sqrt{HE}} \cdot LG$, or $\frac{EL}{CL\sqrt{HC}} = \frac{LI}{LD\sqrt{HE}}$ as before, p. 5.



In Bernoulli's solution the curve between A and B is shewn to be a cycloid; but nothing farther is determined; the result is independent of the relative position



of the points A and B. Hence, the same result, as far as it regards the nature of the curve, will obtain for A and B', A and B'' &c. that is, the curves AB, AB', AB'', &c. are all cycloids. A question then naturally arose out of

the original one; Which is the cycloid, down whose length a body will descend in the least time? Or, in other words, required the curve, a cycloid it must be, such that a body shall descend from A to the vertical line $BB'B''$ in the least time. This, after the solution of the first problem by James Bernoulli, was proposed by him to his brother, and solved by the latter, not only for the particular case in which $BB'B''$ is vertical, but for any inclination of that line, and for any form of that line; that is, John Bernoulli determined the condition to which the cycloid must be subject, when the time down it from A to a curve $b\,b'\,b''$ shall be a minimum. The condition, which hereafter will be deduced, is, that the cycloid shall cut the curve at right angles*.

We may now, without either haste or abruptness, proceed in the historical and scientific development of the new calculus. And, the next step will be to the famous Programma of James Bernoulli, which contained the problem, whence the title of Isoperimetrical, since applied to all problems of a like kind, is derived.

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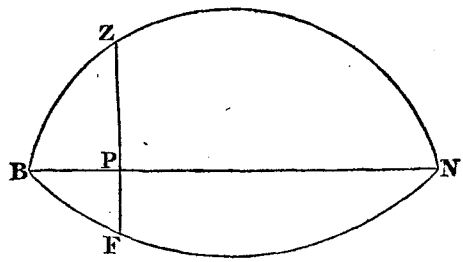
* This Bernoulli determined by an ingenious, but peculiar method. Indeed, there appeared no general method of solving isoperimetrical problems, with all their circumstances, till Lagrange in 1760 gave his formula of solution, consisting of two parts; the one containing integrals, the other a definite expression. See the subsequent parts of this Work.

CHAP. II.

Isoperimetrical Problems proposed by James Bernoulli — Solved by John Bernoulli—His fundamental and specific Equations — Application of them to the Curve of Quickest Descent, and of a given Length—Solution of Isoperimetrical Problems, by Brook Taylor—Imperfections of his and the Bernoullis's Methods.

In 1697, James Bernoulli edited a Programma, which proposed this problem *:

Of all Isoperimetrical curves described on the common base *BN*, to find *BFN* such, that another curve *BZN*



shall contain the greatest space, the ordinate of which *PZ*, is in any multiplicate or submultiplicate ratio of the ordinate *PF*, or of the arc *BF*.

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* "Quæritur ex omnibus isoperimetris, super communi basi *BN* constitutis, illa *BFN*, quæ non ipsa quidem maximum comprehendat spatium, sed faciat, ut aliud curva *BZN* comprehensum sit maximum, cujus applicata *PZ* ponitur esse in ratione quavis multiplicata, vel submultiplicata, rectæ *PF*, vel arcus *BF*, hoc est, quæ sit quotacunque proportionalis ad datam *Ag* rectam *PF*, curvamve *BF*." *Acta Erud.* 1697. Mai. p. 214.

In the terms of modern analysis, the ordinate PZ is said to be a function of PF, or a function of the arc BF.

It is the last case of the problem, when PZ is a function of the arc BF, that baffled the sagacity and skill of John Bernoulli, and caused between him and his brother, a long and acrimonious controversy. All the solutions of the former were erroneous *, till the publication of his brother's. And, even when John Bernoulli had amended his solution, he would not frankly and plainly acknowledge his error, but slurred it over, with a faint and half-confession of having previously been guilty of some slight inadvertencies. The last solution, however, of John Bernoulli, published in the Academy of Sciences for the year 1718, is, considering what was then the state of analytical science, very admirable, and merits the eulogium which he himself has conferred on it, that of being equally exempt from the tediousness of his brother's, and the obscurity of Taylor's calculations †. It is, how-

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* *Acad. des Sciences,* 1706, p. 235. or *Opera,* tom. I. p. 424. In this memoir he solved the first case of his brother's problem, making two elements only of the curve to vary, but making besides the elements of the abscissa to vary. On these grounds the solution of the first case is right, but that of the second erroneous.

† "Une voye courte, claire et facile, suivant laquelle un geometre d'habiletè et d'esprit mediocres puisse arriver jusqu' avoir, non sur la foi d'autrui, mais de ses propres yeux, ces veritez abstruses, sans s'engager dans la longueur du calcul de mon frère, ni dans l'obscurité de celui de M. Taylor." *Opera,* tom. II. p. 238, also *Mem. de l'Acad. de Paris,* 1718. p. 100.

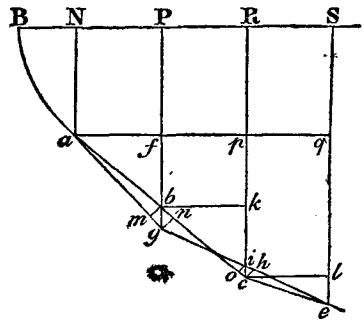
ever, obviously borrowed from his brother's, which was entitled "Analysis magni Problematis Isoperimetrici," and published in the *Acta Eruditorum Lips.* for May, 1701, p. 213. Previously to the *Analysis*, a solution, as it was called, but in which results alone were stated, had been published in the above mentioned Acts, for June, 1700, p. 261.

As this treatise is not intended to be strictly historical, the problem of the isoperimetricals will not be traced through all its variety of change and emendation. To history we shall adhere no farther, than is sufficient to preserve an unbroken series of methods gradually becoming more exact and extensive; the series beginning with the first rude, though perfectly just, method of James Bernoulli, and ending with Lagrange's exquisite and refined *Calculus of Variations*. For this reason we proceed to notice, and shall give the substance of the last solution of John Bernoulli, published in the Memoirs of the Academy of Sciences for 1718. The separation between that, and the first solution of the *brachystochrone*, which has been already given, being neither too wide, nor too abrupt, for our present purpose.

Bae is a curve, and an element of it *abce* is composed of three portions *ab*, *bc*, *ce*; another element *aghe*, indefinitely near the former, is composed of three portions, *ag*, *gi*, *ie*, and since the solution is to extend to *isoperimetrical* curves, we have,

$$ab + bc + ce = ag + gi + ie,$$

and consequently, $(ag - ab) + (gi - bc) + (ie - ce) = 0$



or, $gm - bn - oc + ih = 0$ [1]

By similar triangles,

$$\begin{cases} bmg,\ abf, & gm = \dfrac{fb}{ab} \cdot bg \\ bng,\ bck, & bn = \dfrac{kc}{bc} \cdot bg \\ ioc,\ bck, & co = \dfrac{kc}{bc} \cdot ci. \\ ich,\ cle, & ih = \dfrac{le}{ce} \cdot ci. \end{cases}$$

Hence, substituting in the equation [1]

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right) bg - \left(\frac{kc}{bc} - \frac{le}{ce}\right) ci = 0,$$

and this Bernoulli calls his fundamental equation, on account of the uniformity * subsisting in the coefficients

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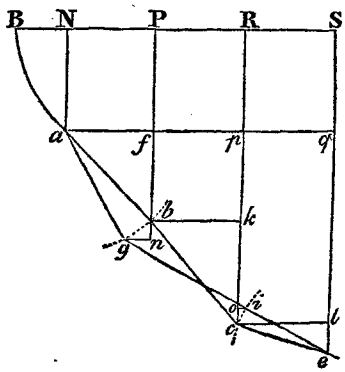
* P. 104. *Mem. Acad. Paris*, 1718. Bernoulli, after noting this uniformity and shewing what it consists in, adds, "On verra dans la suite que cette uniformité contribue merveilleusement a reconnoitre comme d'un seul coup d'œil et non en consequence d'aucune analyse, les equations qui conviennent a chacun des problêmes que nous allons resoudre."

of bg and ci; which uniformity, if we run a little before our matter, and borrow the symbols of the differential calculus, may be easily made manifest. Thus, fb, kc, le, being the differentials of the ordinates Na, Pb, Rc $[y, y', y'']$ are to be noted, dy, dy', dy''; and if dz, dz', dz'' denote the differentials of the three arcs Ba, Bb, Bc, we have,

$$\left(\frac{dy}{dz} - \frac{dy'}{dz'}\right) bg = \left(\frac{dy'}{dz'} - \frac{dy''}{dz''}\right) ci.$$

In the foregoing demonstration, the elements of the abscissa are supposed to be constant, or the points b, c, by whose motion the curve is supposed to be changed, move along the lines fg, pc; but, instead of the elements of the abscissa, those of the curve may be supposed invariable, and this case Bernoulli includes in a second fundamental equation, like the former, and demonstrated by the subjoined process.

Since the elements ab, bc, ec, are supposed to be



constant, from a, e, as centers describe circular arcs

bg, ci, in which, the points b, c, when we pass from curve to curve, are supposed to move: now,

$$bc^2 = gi^2$$

$$\therefore bk^2 + kc^2 = (bk+gn+oi)^2+(kc-co-bn)^2$$

$$\text{or, } (bk+gn+oi)^2 - bk^2 = kc^2 - (kc-co-bn)^2$$

or, by virtue of the formula,

$$x^2 - y^2 = (x+y)\ (x-y)$$

$$(2bk+gn+oi)\ (gn+oi) = [2kc-(co+bn)]\ (co+bn)$$

or, $2bk\ (gn+oi) = 2kc\ (co+bn)$ neglecting $(gn+oi)^2$, $(co+bn)^2$; hence, $gn+oi = (co+bn)\ \frac{kc}{bk}$ . . . [2]

now, by similar triangles, $\begin{cases} afb, bng, & gn = \frac{fb}{af}.bn \\ cle, coi, & oi = \frac{le}{cl}.co. \end{cases}$

Hence, substituting in equation [2], and transposing

$$\left(\frac{fb}{af} - \frac{kc}{bk}\right) bn - \left(\frac{kc}{bk} - \frac{le}{cl}\right).co = 0,$$

which Bernoulli calls also a fundamental equation, and in which is to be noted an uniformity similar to that pointed out in the former fundamental equation.

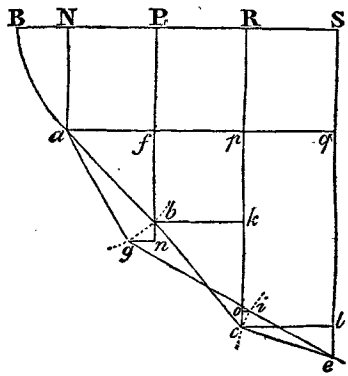
If we express the equation symbolically, we shall have,

$$\left(\frac{dy}{dx} - \frac{dy'}{dx'}\right) bn - \left(\frac{dy'}{dx'} - \frac{dy''}{dx''}\right).co = 0.$$

These equations appeared in Bernoulli's last solution of the Isoperimetrical Problems. And to be satisfied of the skill and previous labour expended in their construction, we need only remark, that they have no

dependance on any property of maximum, or minimum, and seem therefore to have no relation to the problems, for which they were specially designed. The relation, however, which they are made to have, cannot be better understood than from the following instances. The first of which, is the *Brachystochrone*, not the common case*, but that, in which, to the minimum property, the Isoperimetrical is added. In other words, when, out of all curves of a given length, the curve of quickest descent is required.

In the subjoined figure let *Babce* be the curve of



quickest descent, of which a portion *abce* is formed of three elements, *ab, bc, ce*; take a portion of another curve formed of *ag, gi, ie*, indefinitely near the former, then, by the principle of the maxima and minima of quantities, the time down *ab, bc, ce*, = time down *ag, gi, ie*. But the time varies as the space divided by the velocity varies; therefore as the element of the

* See pages 4, and 9.

curve divided by the square root of the vertical height from which the body has fallen: hence,

$$\frac{ab}{\sqrt{Na}}+\frac{bc}{\sqrt{Pb}}+\frac{ce}{\sqrt{Rc}}=\frac{ag}{\sqrt{Na}}+\frac{gi}{\sqrt{Pn}}+\frac{ie}{\sqrt{Ro}}$$

but ab, ag, bc, &c. are all equal;

$$\therefore \frac{1}{\sqrt{Pb}}+\frac{1}{\sqrt{Rc}}=\frac{1}{\sqrt{Pn}}+\frac{1}{\sqrt{Ro}},$$

$$\text{or, } \frac{1}{\sqrt{Pb}}-\frac{1}{\sqrt{Pn}}=\frac{1}{\sqrt{Ro}}-\frac{1}{\sqrt{Rc}},$$

$$\text{or, } \frac{1}{\sqrt{Pb}}-\frac{1}{\sqrt{(Pb+bn)}}=\frac{1}{\sqrt{(Rc-co)}}-\frac{1}{\sqrt{Rc}},$$

$$\text{or, } \frac{1}{\sqrt{Pb}}-\left(\frac{1}{\sqrt{Pb}}-\frac{bn}{2Pb^{\frac{3}{2}}}\right)=\frac{1}{\sqrt{Rc}}+\frac{co}{2Rc^{\frac{3}{2}}}-\frac{1}{\sqrt{Rc}}$$

[expanding and neglecting the terms involving bn^2, co^2, &c.]

Hence, $\frac{bn}{co}=\frac{Pb^{\frac{3}{2}}}{Rc^{\frac{3}{2}}}$; and this is an equation expressing the proportion of bn to co, obtained from the property of the proposed minimum; but the second fundamental equation, involves bn, co, the proportion of bn to co, there arising from the *Isoperimetrical* condition. Hence, eliminating bn, co, there results the specific equation*

$$\left(\frac{fb}{af}-\frac{kc}{bk}\right)Pb^{\frac{3}{2}}-\left(\frac{kc}{bk}-\frac{le}{lc}\right)Rc^{\frac{3}{2}}=0,$$

$$\text{or, } \left(\frac{fb}{af}-\frac{kc}{bk}\right)Pb^{\frac{3}{2}}=\left(\frac{kc}{bk}-\frac{le}{lc}\right)Rc^{\frac{3}{2}}$$

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* "Specifique; parceque d'elle resulte l'equation differentielle, qui determine finalement l'espece de la courbe cherchée." *Acad. des Sciences*, 1718, p. 108.

Now if we observe this equation, it will appear perfectly uniform in its two members, since the right hand member differs from the left, only inasmuch as the points e, c, are used instead of the c, b. Hence, either member is always the same, throughout the curve required: hence, symbolically,

$$\left(\frac{dy}{dx}-\frac{dy'}{dx'}\right)y^{\frac{3}{2}} = \text{a constant quantity};$$

or, $$-d\left(\frac{dy}{dx}\right)y^{\frac{3}{2}} = \surd a\,.\,dz$$

[dz is constant and multiplied into $\surd a$, in order to make the equation uniform.]

that is, $$\frac{d^2x\,.\,dy-d^2y\,.\,dx}{dx^2} = \surd a\,.\,\frac{dz}{y^{\frac{3}{2}}}\;.\;.\;.\;[a]$$

but $dz^2 = dx^2+dy^2$; and since dz is invariable, $o = dx\,.\,d^2x + dy\,d^2y$; hence, substituting in $[a]$

$$\frac{d^2x\,(dx^2+dy^2)}{dx^2} = \surd a\,.\,\frac{dz\,.\,dy}{y^{\frac{3}{2}}}$$

or, $\frac{d^2x}{dx^2}\,.\,dz = \surd a\,\frac{dy}{y^{\frac{3}{2}}}$, or $d\left(\frac{1}{dx}\right).dz = 2\surd a\,.\,d\left(\frac{1}{y^{\frac{1}{2}}}\right)$

and integrating, $\frac{dz}{dx} = 2\sqrt{\frac{a}{y}} \pm \frac{b}{c}$ $\left[\frac{b}{c}\right.$ being the correc^{n}*.]

Hence, $\frac{dz^2}{dx^2}$, or $\frac{dx^2+dy^2}{dx^2} = 4\frac{a}{y} \pm 4\frac{b}{c}\sqrt{\frac{a}{y}+\frac{b^2}{c^2}}$

and, $$dx = \frac{cy^{\frac{1}{2}}\,.\,dy}{\surd(4ac^2+b^2y-c^2y\pm 4bc\surd ay)}.$$

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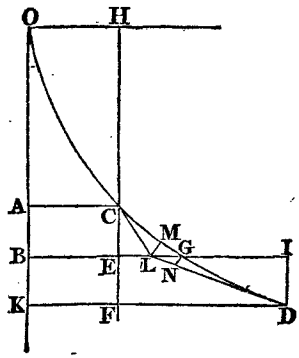
* See Emerson's Fluxions, p. 189. Third Edition.

If in the correction $\frac{b}{c}$, $b = 0$, $dz = 2dx\sqrt{\frac{a}{y}} = 2\sqrt{a}.\frac{dy}{\sqrt{(4a-y)}}$, and the curve is the common cycloid; which now is the curve of the quickest descent, not amongst all curves of the same length between B and C, but amongst all curves whatever that can be drawn between these two points. This is, however, only a particular case, for if b be real, the curve is not a cycloid, nor is it any known curve, but merely that which the differential equation determines it to be.

If we now examine the preceding solution, we shall perceive the office of the fundamental equations in solving problems of maxima and minima. Analytically speaking, they half resolve such problems; and furnish one equation involving two arbitrary quantities, $[bn, co]$ whilst the maximum or minimum property furnishes another. That which results from elimination is the equation of solution.

We may perceive also, in the preceding solution, the necessity of making three elements of the curve to vary. Two elements are sufficient when merely the *brachystochrone* without any restriction is required. The minimum property furnishes one equation, containing an arbitrary quantity $[LG]$, which is also the equation of solution. If a second property, isoperimetrical, or other, be introduced, it may be incompatible with the curve determined by the equation resulting from the minimum property.

For instance, CGD may be determined such, that it



is the curve of quickest descent, and then taking CLD indefinitely near, the time down CG+the time down GD=the time down CL + the time down LD; but $CG+GD$ cannot equal $CL+LD$, for the former sum is less than that of any two lines, drawn from the extremities C, D, externally to CG, GD.

This does not shew, beyond controversy, the necessity of the variation of three elements. For, the points L, G, need not lie in the same line ELG, since the elements of the abscissa as well as those of the curve may be made to vary: in which case, two equations, derived from two properties may be obtained; and this method, which was used by John Bernoulli *, in certain cases, but not generally, leads to right results.

If problems involving merely one property, the

* The method is right in the first case of the problem proposed by James Bernoulli, but not, in the second. See p. 13.

maximum, require the variation of two, and those involving two properties, the variation of three elements; problems involving three properties, would, it should seem, require the variation of four elements: and this is the case. Each property furnishes an equation containing three arbitrary quantities (such as *bn*, *co*), and the solution depends on the equation resulting from their elimination. To such problems, however, the Bernoullis did not extend their researches.

We shall give a second instance of Bernoulli's method in solving the first Isoperimetrical problem that was proposed by his brother; and proceed as in the former case; that is, use two fundamental equations involving *bg*, *ic*, one derived from the Isoperimetrical condition, the other from the common property belonging to quantities, at their state of maximum. Now the problem requires the curve to be determined, such, that if y be the ordinate, and $Y = f(y)$*, $\int Y\,dx$ shall be a maximum.

Hence, since the elements of the abscissa are supposed constant, [see fig. p. 15.]

$$f(Pb) + f(Rc) = f(Pg) + f(Ri),$$

by the property of maximum,

or, $f(Pb) - f(Pg) = f(Ri) - f(Rc)$

or, $f(Pb) - f(Pb + bg) = f(Rc - ci) - f(Rc)$

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* $f(y)$ is the method of noting the function of a quantity y. The name and symbol were used by John Bernoulli in the Memoirs of the Academy for 1706, and 1718. See the latter Vol. p. 106.

from which, expanding $f(Pb+bg)$, $f(Rc-ci)$*, and neglecting the terms involving the squares, cubes, &c. of bg, ci, we have,

$$\text{D}.f(Pb).bg = \text{D}.fRc.ci,$$

[D.$f(Pb)$ being the coefficient of the second term, in the expansion of $f(Pb+bg)$†]

If we now substitute for bg, ci, in the fundamental equation [1] p. 15, there results,

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right)\frac{1}{\text{D}.f(Pb)} = \left(\frac{kc}{bc} - \frac{le}{ce}\right)\frac{1}{\text{D}.f.(Rc)}.$$

In which equation, (specific Bernoulli calls it) the law of uniformity being observable in the composition of its members, it follows, that,

$\left(\frac{fb}{ab} - \frac{kc}{bc}\right)\frac{1}{\text{D}.f(Pb)}$, is every where the same,

or, $\left(\frac{dy}{dz} - \frac{dy'}{dz'}\right)\frac{1}{\text{D}.Y}$, or $-d\left(\frac{dy}{dz}\right)\frac{1}{\text{D}.Y} = \frac{dx}{a}$,

$\frac{dx}{a}$ being from the invariability of the elements of the abscissa, constant;

$$\text{hence, } \frac{d^2z\,dy - d^2y.dz}{dz^2} = \frac{\text{D}.Y.dx}{a},$$

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* $Pb=y$, and $bg=dy$, then, by Taylor's theorem, [See *Princ. Anal. Calc.* p. 44.]

$$f(y+dy) = fy + \frac{d.fy}{dy}.dy + \frac{d^2.fy}{2.dy^2}.dy^2 + \&c.$$

or, if we make $Y=fy$,

$$f(y+dy) = Y + \frac{dY}{dy}.dy + \frac{d^2Y}{2.dy}.dy^2 + \&c.$$

† *Princ. Anal. Calc.* Pref. p. xxix; and p. 43.

but $dz^2 = dx^2 + dy^2$; $\therefore dz \,.\, d^2z = dy \,.\, d^2y$; since dx is constant, consequently, by substitution,

$$\frac{d^2z\,(dy^2 - dz^2)}{dz^2} = \frac{\mathrm{D}\,.\,Y\,.\,dy\,.\,dx}{a},$$

or, $-\frac{d^2z}{dz^2}\,.\,dx = \frac{\mathrm{D}\,.\,Y\,.\,dy}{a}$, and integrating

$\frac{dx}{dz} = \frac{Y \pm c}{a}$, since $Y = \int \mathrm{D}\,.\,Y\,.\,dy = \int \frac{dY}{dy}\,dy = \int dY$, and $\frac{c}{a}$ is the correction; hence,

$$\frac{dx^2}{dx^2 + dy^2} = \frac{(Y \pm c)^2}{a^2}, \text{ and}$$

$$dx = \frac{(Y \pm c)\,.\,dy}{\sqrt{(a^2 - [Y \pm c]^2)}}.$$

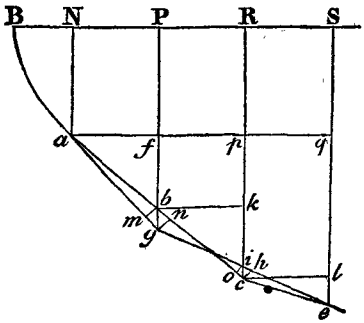
If $Y = f(y) = y$, then $\mathrm{D}\,.\,Y = \frac{dY}{dy} = \frac{dy}{dy} = 1$, and

$$\int dx = \int \frac{(y \pm c)\,.\,dy}{\sqrt{(a^2 - [y \pm c]^2)}},$$

$$\text{or } x = C - \sqrt{(a^2 - [y \pm c]^2)}$$

an equation to a circle. Hence, of all curves, of a given length, that can be drawn between two points, the circle is the curve that contains the greatest area.

If we apply Bernoulli's fundamental equations to the



second case of the Isoperimetrical problem, [see p. 13.]

that is, when the ordinates are not functions of the ordinates Pb, Rc, &c., but of the arcs Bab, $Babc$, &c., the process will be somewhat different from the preceding. For then we have, instead of

$$f(Pb)+f(Rc)=f(Pg)+f(Ri)$$

$$f(Bab)+f(Babc)=f(Bag)+f(Bagi)$$

or, $f(Bab)-f(Bab+gm)=f(Babc-co)-f(Babc)$

and, as before, $\mathrm{D}.f(Bab).gm=\mathrm{D}.f(Babc).co$

But the fundamental equation involves bg, ci; therefore, gm, co, must be expressed in terms of them. Now, by similar triangles,

$$gm=\frac{fb}{ab}.bg,\quad co=\frac{le}{ce}.ci;\text{ hence}$$

$$\mathrm{D}.f(Bab).\frac{fb}{ab}.bg=\mathrm{D}.f(Babc).\frac{le}{ce}.ci;$$

and substituting in the fundamental equation [p. 15. l. 9.]

$$\left(\frac{fb}{ab}-\frac{kc}{bc}\right).\frac{ab}{fb.\mathrm{D}.f(Bab)}=\left(\frac{kc}{bc}-\frac{le}{ce}\right).\frac{ce}{le.\mathrm{D}.f(Babc)}.$$

Now, Bernoulli remarks, that in this equation, as it stands, there is not that uniformity, which enables us to pass from one element ab to another bc affected in the same manner, and so on. There would have been the requisite uniformity, if the factor in the right hand side of the equation, instead of $\frac{ce}{le}$, had been $\frac{bc}{kc}$. We shall cause therefore the requisite uniformity, if we multiply each side by $\frac{bc}{kc}$, and there results the *specific* * equation,

*See Note, p. 19.

$$\left(\frac{fb}{ab} - \frac{kc}{bc}\right) \cdot \frac{ab}{fb} \cdot \frac{bc}{kc} \cdot \frac{1}{\text{D}f(Bab)} = \left(\frac{kc}{bc} - \frac{le}{ce}\right) \cdot \frac{bc}{kc} \cdot$$

$\frac{ce}{le} \cdot \frac{1}{\text{D}.f(Babc)}$ = a constant quantity; hence, since $\frac{ab}{fb} = \frac{bc}{kc}$, and calling $\text{D}.f(Bab) = S$, we have,

$$-d.\left(\frac{dy}{dz}\right) \cdot \frac{dz^2}{dy^2.S} = \frac{dx}{a},$$

or $\frac{d^2z.dy - d^2y.dz}{dz^2} = \frac{S}{a} \cdot \frac{dy^2}{dz^2}.dx$; but, $d^2y.dy = d^2z.dz$;

$$\therefore -\frac{d^2y.dx^2}{dz^3} = \frac{S}{a} \cdot \frac{dy^2}{dz^2}.dx, \text{ or } -\frac{d^2y}{dy^2}.dx = \frac{S.dz}{a};$$

Hence, $\frac{dx}{dy} = \frac{1}{a}(\int S.dz + c) = \frac{Z+c}{a}$, if $Z = \int S.dz$,

or, $a.dx = dy\,(Z+c)$, the equation to the curve.

This, as we have before remarked, is, in the history of Isoperimetrical Problems and their connected Calculus, the most important problem; since, as long as John Bernoulli made two elements of the curve only to vary, he constantly, by his erroneous solutions, afforded matter of triumph to his brother James. The above solution is John Bernoulli's last solution, and an exact one; but, it was not published till 1718, in the Memoirs of the Academy of Paris for that year, sixteen years after the appearance of James Bernoulli's *.

* This solution appeared in the *Acta Erud. Lips.* Mai, 1701, p. 213. John Bernoulli's solution (which was faulty) was communicated to Leibnitz in 1698, to the Academy of Sciences in 1701, and published in their Memoirs for 1706.

On this solution, that of John Bernoulli's is essentially founded; and indeed on the very principle, the neglect of which had vitiated all his former solutions; containing also, under the form of fundamental equations*, James Bernoulli's proportions †. The latter are, with the slightest trouble, transmutable into the former. Yet John Bernoulli, with a total disregard of justice, wishes to confer superiority on his method, by the reverse operation; that is, by transforming the former into the latter. This, however, is not the sole nor the least trait of his unfairness.‡

John Bernoulli's solution possesses greater elegance and compactness than his brother's. Yet these are qualities which we may attribute, almost with equal justice, as well to time as to genius. It possesses, however, a characteristic excellence in the uniformity of its *specific* equations. A considerable advancement was thereby made in the calculus of Isoperimetrical problems; and it is on a like principle of uniformity, that cer-

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* *Mem. Acad.* 1718, p. 103, &c.

† *Acta Erud. Lips.* 1701, p. 213, or James Bernoulli's Works, tom. II. p. 899, or John Bernoulli's Works, tom. II. p. 220, &c.

‡ The palpable error of his former solution he wishes to reduce to a mere fault of inadvertence. "Pour reparer cette faute d'inadvertence," he says, "Je vas donner ici une nouvelle manière de resoudre, &c." Besides, he is ever seeking occasion of aspersing his brother's method, [see *Mem. Acad.* 1718, pp. 102, 103, 131.] and this he does sixteen years after James Bernoulli's death: when, that event, the lapse of time, the recollection of his brother's kindness, a zeal for a brother's fame, ought to have assuaged and laid to sleep all angry passions.

tain methods of solution, of more recent date, are founded*.

Some of the latter assertions the Student must be content to take on trust: for their proof would lead into too wide a digression. Yet, his curiosity will be amply rewarded, if he will search for, what may be found, the identity of the methods employed by James Bernoulli in his "Solutio magni Problematis Isoperimetrici," and by his brother in his last essay, inserted in the Memoirs of the Academy of Sciences.

With the above-mentioned methods, the researches of the Bernoullis on these subjects terminated. Towards the period of their close, in 1715, Brook Taylor, in his "Methodus Incrementorum," solved the problem of the Isoperimetricals, on principles not different from those of the Bernoullis, but with some alteration of symbolical notation. The most material alteration, or rather improvement, consisted in representing the fluxion of V

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* In Lyons's *Fluxions* (1758), at page 99, the author, in solving the *Brachystochrone*, arrives at this equation,

$$\frac{\dot{y}}{V\sqrt{(\dot{x}^2+\dot{y}^2)}} = \frac{\dot{w}}{v\sqrt{(\dot{x}^2+\dot{w}^2)}};$$

in which $\dot{y}$, $\dot{w}$, are the fluxions of two contiguous ordinates, and V, v, the velocities in the arcs. Hence, since there is in this equation a law of uniformity, precisely of the same nature as that which Bernoulli pointed out; the author infers, that $\frac{\dot{y}}{V\dot{t}}$ ($\dot{t}$ the fluxion of the arc) is a given quantity.

The same principle of uniformity serves also, in part, as the basis of Mr. Vince's solutions, *Fluxions*, p. 194, and of some of Emerson's, pp. 182, 183, &c. Third edition.

when $\int V\dot{x}$ is the analytical expression of the maximum property; thus,

$$\dot{V} = M\dot{x} + N\dot{y} + L\dot{s}.$$

which mode of expression, Euler, as we shall hereafter see, skilfully availed himself of.

The methods of the Bernoullis and of Taylor, were held, at the time of their invention, to be most complete and exact. Several imperfections, however, belong to them. They do not apply to problems involving three or more properties; nor do they extend to cases involving differentials of a higher order than the first: for instance, they will not solve the problem, in which a curve is required, that, with its radius of curvature and evolute shall contain the least area. Secondly, they do not extend to cases, in which the analytical expression contains, besides x, y, and their differentials, integral expressions; for instance, they will not solve the second case proposed in James Bernoulli's Programma [see p. 12.] if the Isoperimetrical condition be excluded; for then the arc s, an integral, since it $= \int dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$, is not given. Thirdly, they do not extend to cases, in which the differential function expressing the maximum should depend on a quantity, not given except under the form of a differential equation, and that not integrable; for instance, they will not solve the case of the curve of the quickest descent, in a resisting medium, the descending body being solicited by any forces whatever.

There are no other peculiarities in Taylor's method, than those we have mentioned, that demand our attention. It has no recommendation from its neatness and perspicuity, but is justly censured by John Bernoulli for its obscure conciseness *.

We must now direct our attention to a period of greater interest, during which, the great Euler, who left no part of science untouched or unadorned, directed his attention to the calculus of Isoperimetrical problems.

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* "M. Taylor, homme d'esprit, et Geometre tres habile, qui a heureusement pénétré jusque dans ce que nous avons de plus profond, comme il parôit par son livre de Methodo Incrementorum: sentant bien la longueur embarassante de l'analyse de mon frère, et voulant la rendre plus courte et un peu claire, a repandu lui-même tant d'obscurité sur cette matiere (aussi bien sur d'autres où il a voulu etre court) qu'il semble y prendre plaisir, et que je doute qu'il y ait quelqu'un, quelque pènètrant qu'il soit, qui l'entende partout, quand même la matiere lui seroit dèja connue d'ailleurs," &c. *Mem. Acad.* 1718, p. 103.

CHAP. III.

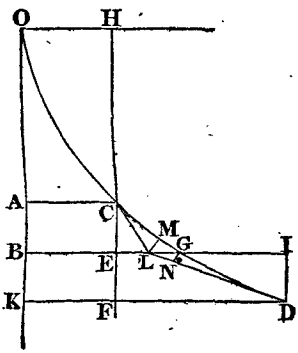
Euler's first Memoir on Isoperimetrical Problems — Table of Formulæ — Solution of Problems by it — Methods of Thomas Simpson, Emerson, and Mac Laurin.

An interval of fifteen years had elapsed, before the subject of Isoperimetrical problems was resumed by Euler. He took it up where the Bernoullis had left it, and conducted his first investigations on their plan. These are not easily intelligible without a previous knowledge of the researches of his great predecessors. The Student, who, for initiation into the peculiar calculus of Isoperimetrical problems, should resort to Euler's first memoir, would, from the novelty of its terms, principles, and methods, find it perplexed and abrupt. But, he will fancy himself gliding along the same route, if after Bernoulli, he takes Euler as the guide of his enquiries*.

Euler's first memoir was inserted in vol. VI. of the Ancient Commentaries of Petersburg for 1733. He there distributes his problems into classes. In

* It is thus with other subjects. Investigation is easy when it is made with the proper series of steps; but difficult, when by our own sagacity and labour, we must supply either steps that are wanting, or approximate those which are too widely separate.

the first, are problems like that of the *brachystochrone* and the curve of least resistance, with the property of the minimum, but without the Isoperimetrical condition or any other. These are to be solved from the principle of the property of a maximum, belonging to the elements of the curve, as well as to the curve itself; and from the principle of the equality between two proximate states of a quantity, when near its minimum or maximum *; and they require for their solution the variation of two elements only of the curve. Thus, in the subjoined Figure which is constructed similarly to Euler's †, let *CLND* be the element that possesses the



property of a proposed maximum or minimum: take *CMGD* indefinitely near; then Euler says, if we express the property in *CMGD* and in *CLND*, the difference of the two expressions ought to equal 0.

[*a*] Now *CL* is changed into $CG = CM + MG = CL + \frac{LE}{CL}.LG$,

* See p. 7. † *Comm. Acad. Petrop.* tom. VI. p. 128.

[*b*] and LD is changed into $DG = DL - LN = DL - \frac{LI}{LD}.LG$,

or the variations of CL, LD are $\frac{LE}{CL}.\ LG$, $-\ \frac{LI}{DL}.LG$.

Hence, in an instance, in which $\int x^n.ds$, is to be a minimum (x = abscissa, ds = differential of the curve,) we have,

$$OA^n.CL + OB^n.LD = OA^n.CG + OB^n.GD,$$

hence, by [*a*], [*b*],

$$OA^n.\frac{LE}{CL}.LG = OB^n.\frac{LI}{LD}.LG,$$

$$\text{or } OA^n.\frac{LE}{CL} = OB^n.\frac{LI}{LD},$$

and consequently from the *uniformity* of this equation, [p. 15, 20, &c.]

$OA^n.\frac{LE}{CL}$, or $x^n.\frac{dy}{ds} = a$ the equation to the curve.

If $n = -\frac{1}{2}$, $\frac{dy}{ds\sqrt{x}} = a$, the equation to a cycloid, and the expression for the minimum is $\int \frac{ds}{\sqrt{x}}$, or $\int dt$ (t the time,) or the time is a minimum down the curve, whose equation is $\frac{dy}{ds\sqrt{x}} = a$. This problem is that which was solved in pp. 4, 9. and on the same principles; for if we compare the solutions, the two latter will be found to differ from the former, merely in the greater compactness and regularity of their processes, and in

their furnishing something like a clue to the general solution of similar problems*.

The preceding problem is easily generalised; for if P be a function of x, and $\int P ds$ is to be a minimum, we should have, supposing P to become P' when x becomes $x + dx$,

$$P.CL + P'.LD = P.CG + P'.GD$$

and by $[a]$, $[b]$,

$$P.\frac{LE}{CL}.LG = P'.\frac{LI}{LD}.LG, \text{ or } P.\frac{LE}{CL} = P'.\frac{LI}{LD}$$

an equation, which, according to Bernoulli, may be called a specific equation.

Hence, $P.\frac{dy}{ds} = a$, the equation to the curve.

By these methods, hardly to be reckoned different from those of the Bernoullis, Euler solves problems of the first class: he then passes on to those of the second, which besides the property $[B]$ of the maximum, are to have another property $[A]$ †; of this class are those

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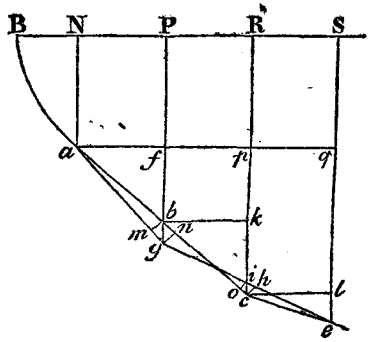
* The first solution of James Bernoulli, as it stands [see p. 4.] affords scarcely the least glimpse of a general method or formula of solution. Of its peculiarity, the Editor of John Bernoulli's Works is aware, since he remarks, "Quoniam autem synthesin meram particularem, nec ad similes casus facile applicandam, &c."

† The following is Euler's classification;
"I. Ex omnibus prorsus curvis eam determinare, quæ proprietatem A maximo vel minimo gradu contineat. II. Ex omnibus curvis proprietate A æqualiter præditis, eam determinare, quæ proprietatem B maximo

problems solved by James and John Bernoulli, in which the property $[A]$, is the Isoperimetrical one. These require the variation of three elements ab, bc, ce;



and from such John Bernoulli procures, in fact, two equations, involving bg, ci: one equation * called a fundamental equation, from the Isoperimetrical property $[A]$; the other † from the property of the maximum $[B]$: and these equations may be thus represented:

$[c] \ldots P.bg - Q.ci = 0$ from Isop^{l}. condit. $[A]$,

$[d] \ldots R.bg - S.ci = 0$ from max^{m}. condit. $[B]$;

whence results $PS = QR$, which, if the law of uni-

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maximo vel minimo gradu contineat. *III.* Ex omnibus curvis et proprietate *A* et proprietate *B* æqualiter præditis eam determinare, quæ proprietatem *C* maximo minimove gradu contineat. *IV.* Ex omnibus curvis proprietatibus *A* et *B* et *C* singulis æqualiter præditis, eam determinare, quæ proprietatem *D* maximo minimove gradu contineat.——Simili modo quinta classis curvas quatuor proprietatibus præditas contemplabitur et ita porro sequentes." *Comm. Acad. Petrop.* tom. VI. p. 125.

* p. 15.

† p. 19.

formity prevailed in it, as in the two first instances *, would be a specific equation: but, if the necessary uniformity should be wanting, then it must be introduced, by multiplying each side of the equation by proper quantities, as was done in the solution of the third problem †. Now Euler's plan and reasoning is very similar to this: he deduces the equations [*c*] [*d*]; and then observes, that the quantities Q and S are frequently so compounded, that $Q = P + dP$, and $S = R + dR$; or, if not so compounded, may, by multiplication, be reduced to that form ‡.

Euler, in this part, gives no general proof of the preceding assertion; but, if it be admitted, then the equation

$$QR = PS, \quad \text{becomes}$$

$$R(P + dP) = P(R + dR), \quad \text{and consequently,}$$

$$R.dP = P.dR, \text{ or } \frac{dR}{R} = \frac{dP}{P}, \text{ and integrating}$$

$$P + aR = 0, \quad \text{the equation to the curve.}$$

In point of practical convenience of solution, a great step was made by Euler in the preceding process. For the computist is directed to attend solely to the deriving, from one of the proposed properties, an equation of the form,

$$P.bg - (P + dP)\,ci;$$

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* pp. 20, 24. † p. 26.

‡ "In quibus quantitates $\mathfrak{Q}$ et S plerumque ita sunt comparatæ, ut sit $\mathfrak{Q} = P + dP$, et $S = R + dR$. Si vero hujusmodi formam non habuerint, poterunt semper multiplicando vel dividendo æquationes ad talem reduci." *Comm. Acad. Petrop.* tom. VI. p. 134.

whence P will be known. He must then derive from the other of the properties a similar equation,

$$R.bg - (R + dR).ci$$

whence R will be known: and the resulting equation of solution will be

$$P + aR.$$

For instance, suppose one of the properties $[B]$, to be such, that $\int y^n.dx$ shall be a maximum; then, since dx is constant,

$$Na^n + Pb^n + Rc^n = Na^n + Pg^n + Ri^n$$

and $\therefore Pb^n - (Pb + bg)^n = (Rc - ci)^n - Rc^n$,

and, $Pb^{n-1}bg = Rc^{n-1}.ci$,

or $Pb^{n-1}.bg - Rc^{n-1}.ci = 0$,

which, since Rc is the contiguous or consecutive ordinate to Pb, and therefore (if $Pb = y$) $= y + dy$, is of the proposed form,

$$P.bg - (P + dP)\,ci,$$

in which $P = y^{n-1} = \frac{d.y^n}{dy}$, or to render the equation homogeneous, $y^{n-1}.dx$; similarly, if T had been a function of y and the property $\int T.dx$; P would have been $= \frac{dT}{dy}$ or $\frac{dT}{dy}.dx$. Suppose the other property $[A]$ to have been $\int x^m.ds = a$, a being constant and ds the element of the curve, then

$$BN^m.ab + BP^m.bc + BR^m.ce = BN^m.ag + BP^m.gi + BR^m.ie,$$

and substituting according to the forms $[a]$, $[b]$, p. 33. there will result

$\left(BN^m.\frac{bf}{ab} - BP^m.\frac{kc}{bc}\right).bg - \left(BP^m.\frac{kc}{bc} - BR^m.\frac{le}{ce}\right).ci$ $= 0$ *, which is evidently under the proposed form, if we make $R = BN^m.\frac{bf}{ab} - BP^m\frac{kc}{bc}$, or symbolically $= d\left(\frac{x^m.dy}{ds}\right)$. Similarly, if instead of $\int x^m\,ds$, the form had been $\int X.ds$, X a function of x, then there would have resulted,

$$R = d\left(\frac{X.dy}{ds}\right).$$

The equation therefore to the curve with these two properties, A and B,

$$\text{is } y^{n-1}.dx + a.d\left(\frac{x^m.dy}{ds}\right)$$

$$\text{or generally, } \frac{dT}{dy}\,dx + a.d\left(X.\frac{dy}{ds}\right).$$

Since the equation $P.bg - (P + dP)\,ci$, derived from the property B, is precisely similar to the equation, $R.bg - (R + dR)\,ci$, derived from the property A; it is plain, that the resulting equation, that which determines the curve, or the relation between y and x, will be the same, if B and A were changed; that is, if

$$P.bg - (P + dP)\,ci$$

be derived from A, and

$$R.bg - (R + dR)\,ci$$

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* This is exactly under the form of John Bernoulli's *specific* equations, for the law of uniformity is manifest in it.

from B. For instance, suppose the curve to be required which, with a given length, should contain the greatest area; here the maximum property B is $\int y\,dx$, and the Isoperimetrical property A, is $\int \sqrt{(dx^2 + dy^2)} = a$. The resulting equation is $dy = \frac{x\,dx}{\sqrt{(r^2 - x^2)}}$ an equation to a circle: which equation would equally result, if we required the curve which, with a *given area*, should contain the *greatest arc*.

This important remark of the commutability of the properties was originally made by James Bernoulli*.

Euler, we have seen, reduced problems of the second class to a dependance on two similar equations of the form

$$P.bg - (P + dP)\,ci = 0,$$

the determination of P depending on the proposed properties: for instance, if the property A or B were $\int T.dx$, $T = f(y)$, P would equal $\frac{dT}{dy}$, or $\frac{dT}{dy}.dx$. If the pro-

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" In quibus omnibus singularis quædam observatur reciprocatio. Quemadmodum enim, exempli gratiâ, inter omnes figuras ejusdem Perimetri circulus maximam possidet aream, catenaria maximam conversione sui gignit superficiem, solidumque maximum elastica; sic inter omnes vicissim figuras quæ aut æqualibus gaudent areis, aut æquales rotatione gignunt superficies, solidave æqualia; circulus, catenària et elastica minimo clauduntur ambitu, quod pariter procedit in omnibus aliis." *Acta Erud. Lips.* Mai, 1701, p. 213. or *Opera*, tom. II. p. 919. or, John Bernoulli, tom. II. p. 234. — Euler makes the same remark. *Comm. Acad. Petrop.* tom. VI. pp. 135, 150. and tom. VIII. p. 175. and in his *Methodus Inveniendi Curvas, &c.* See also Emerson's *Fluxions*, p. 170.

perty were $\int T.dy$, $T=f(x)$, P would $=\frac{dT}{dx}$ or $\frac{dT}{dx}.dx$. If the property were $\int T.ds$, $T=f(x)$, P would equal $d\left(T.\frac{dy}{ds}\right)$; and by observation on the resulting forms for P, Euler generalised his conclusions, and arranged them in a table, after the manner of the subjoined specimen*:

Proprietates propositæ.		Valores Litteræ P respondentes.
I. $\int T.dx$	$dT=Mdy$†	$P=M.dx$
II. $\int T.dy$	$dT=Ndx$	$P=N.dx$
III. $\int T.ds$	$dT=Ndx$	$P=d.\left(T.\frac{dy}{ds}\right)$
IV. $\int T.ds$	$dT=Mdy$	$P=d.\left(T\frac{dy}{ds}\right)-Mds$,

&c.

and of these forms he gave fifteen, by reference to which, any problem belonging to the second class might be solved.

For instance, suppose the curve to be required, which, amongst all others of the same length, should contain the greatest area. Here,

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* *Comm. Acad. Petrop.* tom. VI. p. 141.

† $dT=Mdy$, and $M=\frac{dT}{dy}$, or M is the differential coefficient of T, making in T, y to vary. Similarly, $N=\frac{dT}{dx}$ is the differential coefficient making in T, x to vary. If T should contain both x and y, that is, if $dT=Mdy+Ndx$, then M and N would become partial differential coefficients. See *Princ. Anal. Calc.* p. 79.

the maximum property $B, = \int y\,dx,$

the Isoperimetrical $A, = \int ds = a.$

By Form I. $T = y$; $\therefore M = 1$, $P = dx.$

By Form III. $T = 1$; $\quad P = d\left(\frac{dy}{ds}\right)$ or $R = d\left(\frac{dy}{ds}\right)$

Hence, since the equation is $P \pm aR = 0$ [see p. 38.] $dx \pm a.d\left(\frac{dy}{ds}\right) = 0$, or $x + a\frac{dy}{ds} - c = 0$ (c = correction); and by reduction,

$$dy = \frac{(x-c).dx}{\sqrt{(a^2 - [x-c]^2)}}$$ an equation to a circle *.

Again, suppose the curve to be required, which, amongst all others of the same length, shall, by a rotation round its axis, generate the greatest solid. Here, $B = \int y^2.dx$; $\therefore$ by Form I. $T = y^2$, $M = 2y$, $P = 2y.dx$;

$A = \int ds$; $\quad \therefore$ by Form III. P or $R = d\left(\frac{dy}{ds}\right)$;

Hence, $2y.dx + a.d\left(\frac{dy}{ds}\right) = 0,$

or, $2y.d\dot{x} + a.\frac{ds.d^2y - dy.d^2s}{ds^2} = 0.$

But since dx is constant, and $ds^2 = dx^2 + dy^2$, $ds.d^2s = dy.d^2y$; therefore, substituting,

$$2y.dx + a.\frac{d^2y.dx^2}{ds^3} = 0,$$

* See Emerson's *Fluxions*, third edition, p. 187; also Simpson's *Fluxions*, p. 485.

and, $2y + \frac{a.d^2y.dx}{(dx^2+dy^2)^{\frac{3}{2}}} = 0$; multiply by dy, and integrate, and we have

$$y^2 - \frac{a.dx}{(dx^2+dy^2)^{\frac{1}{2}}} = c \quad [c = \text{correction},]$$

whence, $dx = \frac{(y^2-c)\,dy}{\sqrt{(a^2-[y^2-c]^2)}}$, an equation to the elastic curve; and which in a particular case, when $c=0$, becomes

$$dx = \frac{y^2.dy}{\sqrt{(a^2-y^4)}},$$

and the curve in this case is called the rectangular elastic curve *.

As a third example, let the curve be required, which, amongst all others of the same length, shall have its center of gravity most remote from the axis. Here, (calling x the distance from the axis) $B = \int \frac{x.ds}{s}$; $\therefore$ by Form III. [since s is a given quantity] $P = d\left(x\frac{dy}{ds}\right)$ again $A = \int ds$; $\therefore$ by Form III. P, or $R = d\left(\frac{dy}{ds}\right)$

Hence, $a.d\left(\frac{dy}{ds}\right) + d\left(x\frac{dy}{ds}\right) = 0$;

$\therefore\ (a+x)\frac{dy}{ds} = c$, or $c\,ds = (a+x).dy$

an equation to the catenary.

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* See Simpson's *Fluxions*, p. 486, where the solution is not general.

This example could not have been solved by Euler's table, if the property had been any other than the Isoperimetrical: for s, an integral, $= \int dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$; and Euler gives, in this memoir, no general method of finding the resulting equation, such as P is in his table, when the analytical expression of a property involves integrals. See tom. VI, p. 144.

By means of this table, the practical solution of Isoperimetrical problems, was, as it has been already said, very materially expedited. In a subsequent part of his memoir*, Euler increases his table by nine new forms: making the whole number twenty-four. And although this table is now superseded, yet its examination is not without interest, since we may discover in it the parcels of that general formula, which the author afterwards exhibited.

Having given rules of solution for all problems belonging to the second class, Euler passes on to those of the third. In this a curve is required, which, amongst all others equally possessing the property A and the property B, contains a property C of maximum or minimum. For instance, if the curve be required, which, amongst all curves of the same length and the same area, is such, that the time down it is a minimum. This class requires the variation of four elements of the

* *Comm. Petrop.* tom. VI, p. 146.

curve; and consequently the methods of the Bernoullis, without extension, are insufficient. The course pursued by Euler is like his former one in the second class. He constructs a diagram similar to the one, p. 36, the sole difference arising from the introduction of a fourth element of the curve; and he reduces the analytical solution to a dependence on three similar equations instead of two.

The general form of these equations is,

$$P.bg - Q.ci + R.d\delta$$

$d\delta$* being a variation of the ordinate similar to the variations bg and ci; and he makes an observation similar to the one in p. 37; namely, that it frequently happens in simple cases, that $Q = P + dP$, $R = P + 2dP + d^2P$; but, if such should not be the form of Q and R, then the skill of the analyst must be directed to reduce them under that form.

When three equations, such as

$$P.bg - (P + dP)\,ci + (P + 2dP + d^2P).d\delta = 0$$
$$p.bg - (p + dp)\,ci + (p + 2dp + d^2p).d\delta = 0$$
$$\pi.bg - (\pi + d\pi)\,ci + (\pi + 2d\pi + d^2\pi).d\delta = 0$$

are obtained, the resulting equation to the curve will be†

$$P + ap + b\pi = 0 \quad [1]$$

for, taking the differential, $dP + a.dp + b.d\pi = 0$ [2]

and again, $d^2P + a.a^2p + b.d^2\pi = 0$ [3]

And since, $[1] \times bg = 0$; $([1] + [2]) \times ci = 0$, and $([1] + 2[2] + [3]) \times d\delta = 0$;

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* $d\delta$ has no connexion whatever with the separate symbols d, δ

† Euler, *Comm. Acad. Petrop.* tom. VI, p. 149.

$$\therefore \quad P.bg \quad + \quad ap.bg \quad + \quad b\pi.bg \quad = 0$$
$$-(P + dP).ci - a(p + dp).ci - b(\pi + d\pi).ci = 0$$
$$(P + 2dP + d^2P).d\delta + a(p + 2dp + d^2p)d\delta + b(\pi + 2d\pi + d^2\pi)\, d\delta = 0;$$

which equations, adding the quantities that are placed vertically, verify themselves; and accordingly shew, that the equation

$$P + ap + b\pi = 0$$

satisfies the three equations.

The solution of problems then would be similar to that of those in the second class, and would be immediately obtained, if the quantities P, p, π, were contained in his table of forms *; and in many cases they are. For instance, suppose the curve to be required, which amongst all other curves, of the same length and same area, generates, by a rotation round an axis parallel to the ordinate y, the greatest solid. Here,

the max^{m}. property C is $\int x^2 dy$; $\therefore$ by II. p. 41. $P = 2xdx$,

the . . . property A is $\int y dx$; $\therefore$ by I. $p = dx$,

the . . . property B is $\int ds$; $\therefore$ by III. $\pi = d\left(\frac{dy}{ds}\right)$;

and consequently, the equation to the curve is

$$2xdx + a.dx + b.d\left(\frac{dy}{ds}\right) = 0;$$

and integrating, $x^2 + ax + b\frac{dy}{ds} = c$ [c = correction]

whence $dy = \frac{(x^2 + ax - c)\, dx}{\sqrt{(b^2 - [x^2 + ax - c]^2)}}$.

* *Comm. Petrop.* tom. VI, p. 151.

Again, as a second instance, let us take a third case of the *brachystochrone*, of which we have already had two, pp. 4, 18; that is, besides the condition C of quickest descent, let the conditions A and B be those of a given length, and a given area;

here, $C = \int \frac{ds}{\sqrt{x}}$; $\therefore$ by III. $P = d\left(\frac{dy}{ds\sqrt{x}}\right)$

$A = \int y dx$; $\therefore$ by I. $p = dx$

$B = \int ds$; $\therefore$ by III. $\pi = d\left(\frac{dy}{ds}\right)$

and consequently, the equation to the curve is

$$d\left(\frac{dy}{ds\sqrt{x}}\right) + a dx + b.d\left(\frac{dy}{ds}\right) = 0;$$

and integrating, $\frac{dy}{ds\sqrt{x}} + ax + b\frac{dy}{ds} = c.$

Euler, as we have observed, commenced his researches where the Bernoullis had terminated theirs; but, at the end of this his first investigation, he was considerably removed from the original point of departure. Several important objects had been attained by him; the solution of problems involving three or more properties; the reduction of such problems to a dependence on two or more similar equations; the solution of problems of the first class, and of some of higher classes, by a more general method*, and by reference to a table of formulæ.

These methods were held, by their author, to be so

* Tamen eas methodo paulisper diversa et multo latius patente sum persecuturus, &c. *Comm. Petrop.* tom. VI. p. 127.

complete, that, on the ground of facility, nothing farther was to be desired*. They are, however, not exempt from several imperfections. Problems, involving the differentials of x or y of an higher order than the second cannot be solved by them; for instance, that which requires, amongst all other curves, one, in which $\int \frac{d^3y}{dx.dy}$ is a maximum or minimum. Secondly, problems cannot be generally solved by them which involve integrals, such not being constant: for instance, that in which it should be required to find a curve that amongst all other curves, has its center of gravity lowest. Euler solves this problem when another condition, that of the Isoperimetrical property, is added: for then the arc s, the integral of $dx\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}$, is in all curves supposed to be the same †.

The researches of Maclaurin ‡, Emerson §, and Simpson ||, on this subject, may here be noticed. With regard to practical methods of solution they do not extend so far as those of Euler, which we have been speaking of; and in point of perspicuity, if we except Maclaurin, the other two mathematicians are inferior to the learned foreigner.

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* Atque methodo tam facili solutum ut nihil amplius desiderari posse videatur. *Comm. Petrop.* tom. VIII, p. 159.

† See p. 43.

‡ *Fluxions*, p. 478. § *Fluxions*, Third edition, p. 170.

|| *Fluxions*, vol. II. p. 480. Third edition, 1750.

The methods of Maclaurin and Simpson (for Emerson's is plainly taken from that of the former) extend to cases in which more than one property is involved: but they are inapplicable to the three cases, and the connected problems enumerated in p. 30.

Maclaurin's formula of solution is this: If X and Z are functions of x, then if $Xds - Zdy$ be a minimum or maximum, $Xdy = Zds$. This result is included amongst Euler's. [See *Comm. Acad. Petrop.* tom. VI. pp. 141, 142, 143.] For since Xds expresses one property, and $dX = \frac{dX}{dx}\,dx$, or since X is a function of x, we have, by Form III. the quantity corresponding to P [see p. 41.] $= d\left(X.\frac{dy}{ds}\right)$, and for Zdy expressing the other property, by Form II. the quantity, corresponding to P, $= \frac{dZ}{dx}.dx$; consequently, the resulting equation is

$$d\left(X\frac{dy}{ds}\right) = a.dZ \quad \text{and} \quad Xdy = aZds,$$

the same result in fact as Maclaurin's.

Simpson's method is equally restricted with Maclaurin's: it rests too on the assumption of the principle, that the property of minimum or maximum, true for the whole curve, is true also for any portion of it. The want of generality, therefore, in this principle, would vitiate the method in its application to the excepted cases.

The methods just described solve not problems of greater depth and intricacy than those of the Bernoullis; although, it must be remarked, they are invested with

greater analytical neatness and compactness. They are not, however, more perspicuous; and, even if they did possess greater extent and clearness, it would not suit the purpose of the present Tract, longer to insist on them, since they conduct us not towards that formula and algorithm, with which the researches on this subject have been closed.

The plan of this Work now leads us to the consideration of Euler's second memoir; in which, he very materially improved the calculus of Isoperimetrical problems; although he introduces his resumed researches with a prefatory remark, that his former methods of solution, on the footing of facility, left nothing farther to be desired *.

* Atque methodo tam facili solutum, ut nihil amplius desiderari posse videatur. *Comm. Petrop.* tom. VIII, p. 159.

CHAP. IV.

Euler's second Memoir—General Formulæ of Solution—Characters of distinction, which Problems admit of—Exceptions to the general Formulæ.

THE formation of the table * in Euler's former memoir depended on this principle: If $\int V dx$ is the analytical expression of the maximum property †, certain functions of x, y, &c. are substituted for V, and thence

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* P. 41.

† Since in the succeeding part of this Tract $\int V dx$ will be frequently used as the analytical expression of the maximum property, it may not be improper to illustrate its meaning by one or two examples.

In the *brachystochrone*, the time $= \int \frac{ds}{\sqrt{y}}$ = a minimum; therefore $\int \frac{ds}{\sqrt{y}} = \int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{y}} = \int \frac{dx\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}}{\sqrt{y}}$, which, compared with $\int V dx$, gives $V = \frac{\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}}{\sqrt{y}} = \frac{\sqrt{(1+p^2)}}{\sqrt{y}}$ $\left[p = \frac{dy}{dx}\right]$.

In the solid of least resistance, the resistance $= \int \frac{y.dy^3}{dx^2+dy^2} = \int \frac{y.p^3}{1+p^2}.dx$; therefore, comparing, $V = \frac{yp^3}{1+p^2}$.

If, with a given area, the curve generating the solid of least surface is required, then [see p. 37.] $\int V dx = \int (2\pi y.ds + ay dx) = \int (2\pi y\sqrt{[1+p^2]}.dx + ay\,dx)$; $\therefore V = ay + 2\pi y\sqrt{(1+p^2)}$.

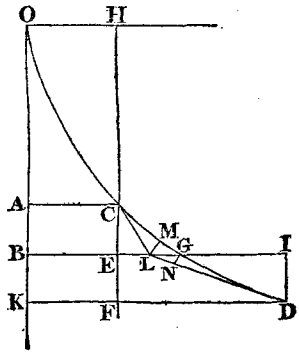
And from these expressions, the value of dV may be found by the common rules of the differential or fluxionary calculus.

a resulting equation (according to Euler the equation $P=0$), is obtained. In the present memoir, he represents the differential of V, or dV, as Taylor had done [p. 30.]; that is, he puts

$$dV = Mdx + Ndy + Pdp + \&c. + Lds^*$$

$$\left[p = \frac{dy}{dx}, \text{ and } s = \text{arc},\right]$$

and from this more general formula of representation, he deduces an equation comprehending almost all his former particular equations which he had registered in a table [p. 41.]. His mode of proceeding is not materially different from his former mode. If V contains no integrals, but is merely a function of x, y, dy, &c., then



the property expressed by $\int V dx$ is common to the curve, and to its elementary portions: hence, if we suppose two portions only of the curve to vary, and if Vdx corresponds to CL, and $V'.dx$ to LD, $(V+V')$

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* P in this and the following pages is merely a coefficient, and different from the P of the preceding Chapter.

dx is a minimum. Hence, that peculiar differential of $(V+V')\,dx$, which arises from changing CL, LD, into CG, GD, is equal to nothing: which differential is easily obtained from the common differential, since each has this in common, that of being the first term of the difference of two successive values: now the common differential,

$$\text{or, } (dV+dV')\,dx =$$

$$(Mdx+Ndy+Pdp)\,.\,dx+(M'dx+N'dy'+P'dp')\,dx,$$

supposing the differential of V to consist of three terms only, and M', N', P', to be the values of M, N, P, when V becomes V'. Now, in order to compute the changes in the several terms, we may observe, that since $AC=y$, and since dx is supposed to be constant, the terms Mdx, Ndy, are not affected by the translation of the point L to G; but Pdp is; for $p=\frac{dy}{dx}=\frac{EL}{CE}$: but when L is transferred to G, $p=\frac{EG}{CE}=\frac{EL+LG}{CE}$; consequently the variation produced in p, or what dp becomes, when it expresses the peculiar differential which we are seeking, is $\frac{LG}{CE}$*. Hence the term Pdp

* Those who are possessed of the calculus of variations will here perceive the very great advantage accruing from the mere invention of a symbol δ to denote an operation, bordering on, yet distinguished from, the operation which d denotes. Circumlocution and ambiguity were both rescinded by that invention. It is wonderful that Euler did not hit on it: for the rules belonging to such a symbol, he plainly lays down in *Comm. Petrop.* tom. VIII, p. 163, and afterwards more

must be written $P.\frac{\nu}{dx}$, putting $\nu = LG$, and $dx = CE$. Again, $y' = BL$, and by the translation of L to G, $= BG = BL + \nu$. Hence, dy' must be written ν, and $N'.dy'$, $N'\nu$. again $p' = \frac{LI}{dx}$, and by the translation of L to $G = \frac{LI - LG}{dx} = \frac{LI - \nu}{dx}$. Hence, dp' must be written $\frac{-\nu}{dx}$, and $P'.dp', \frac{-P'\nu}{dx}$. Collecting the several terms, we have the whole variation of $(V + V')\,dx$ equal to $\left(P.\frac{\nu}{dx} + N'\nu - \frac{P'\nu}{dx}\right).dx$, or $\left(N' - \frac{P' - P}{dx}\right)\nu.dx$, or $\left(N - \frac{dP}{dx}\right)\nu.dx$, retaining quantities of the same order: lastly, since the variation must equal nothing, we have for cases comprehended within the expression $dV = Mdx + Ndy + Pdp$, this formula of solution,

$$N - \frac{dP}{dx} = 0.$$

By a process similar, but longer on account of the introduction of the quantity ds, Euler deduces from

$$dV = Mdx + Ndy + Pdp + Lds$$

this formula of solution,

$$N - \frac{dP}{dx} + \frac{Ldy}{ds} = 0.$$

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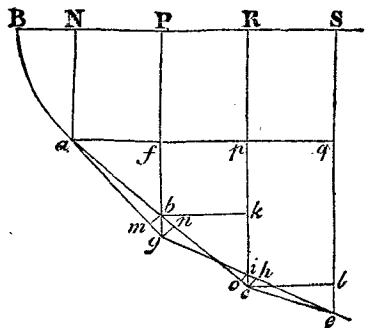
more fully and distinctly in his tract on Isoperimetricals; and in his last memoir, *Novi Comm. Petrop.* tom. X. p. 54, he says, "Illud problema Isoperimetricum latissimo sensu acceptum, prout id quidem in libro singulari pertractavi; quem qui attente legerit, non dubitabit, quin hujus generis investigationes calculi speciem prorsus singularem postulent, a consuetis analyseos regulis non parum diversam."

The two preceding formulæ have been deduced for the first class of problems that involve one property only. But Euler's method, although more tedious, is not essentially different for problems of the second class. These require the variation of three elements of the curve; and accordingly, we must compute the variation in



$(V+V'+V'')\,dx$, when the three elements, instead of ab, bc, ce, become ag, gi, ie; that is, as before we must note the variations in

$$M.dx + N.dy + P.dp + \&c.$$
$$M'.dx + N'.dy' + P'.dp' + \&c.$$
$$M''.dx + N''.dy'' + P''.dp'' + \&c.$$

arising from the translation of the points b and c, to g and i, and from the introduction of two arbitrary quantities or variations such as bg, ci, and thence will result an equation of the form $A.bg - B.ci = 0$, which compared with the equation, $R.bg - (R+dR).ci = 0$, which Euler

had previously obtained *, would give $\frac{dR}{R}=\frac{B-A}{A}$, whence R is known.

In the formula $dV=Mdx+Ndy+Pdp$ (forming the equation $A.bg-B.ci$, according to the precepts just given,)

$$A = N'.dx - dP$$
$$B = N''.dx - dP'$$

consequently, $\frac{dR}{R}=\frac{d(N'dx-dP)}{N'dx-dP}$, and $R=N'dx-dP$,

or $= dx\left(N-\frac{dP}{dx}\right)$.

This is an equation derived from one property $\int Vdx$: deduce a similar equation from the other $\int Wdx$; for instance, such as $\varrho=dx\left(\nu-\frac{d\pi}{dx}\right)$, and the equation to the curve will be

$$R+a\varrho=0, \text{ or } \left(N-\frac{dP}{dx}\right)+a\left(\nu-\frac{d\pi}{dx}\right)=0;$$

$$\text{or } N+a\nu-\frac{d(P+a\pi)}{dx}=0;$$

and since this form is like that of $N-\frac{dP}{dx}$, instead of two operations for finding the values of R, R', from $\int Vdx=$ a maximum, and $\int Wdx=$ a constant quantity, we may substitute one, and deduce the resulting equation from

* P. 37.

$$\int V dx + a \int W dx,$$
$$\text{or } \int (V + aW)\, dx,$$

and for this great simplicity introduced into the calculus, we are indebted solely to Euler.

For the sake of stating and illustrating Euler's method, we have taken a simple form to represent dV, that is, $Mdx+Ndy+Pdp$: but Euler introduces a term Lds, which, for problems of the second class, renders the calculation longer, and the result more complicated: in this case, however,

$$\frac{dR}{R} = \frac{-d^2P + L'dx.d\varepsilon + dx d(L'\varepsilon + N)}{-dP + dx\,(L'\varepsilon + N)} \quad \left[\varepsilon = \frac{dy}{ds}\right]$$
$$= \frac{L'dx.d\varepsilon}{-dP + dx(L'\varepsilon + N)} + \frac{d[-dP + dx(L'\varepsilon + N)]}{-dP + dx(L'\varepsilon + N)};$$

and, $R = e^{\frac{\int L dx.d\varepsilon}{-dP+dx(L\varepsilon+N)}} \times (-dP + dx\ [L\varepsilon + M])$.

This memoir of Euler contains almost all the matter which is to be found in his subsequent tract on Isoperimetricals: but, the matter is ill arranged: considerable perplexity, and some error, is introduced by the term Lds: the cases in which it enters, are mixed with those in which $L=0$. These latter, in the above mentioned tract, the author separately considers; and even here he notices, that as from $dV=Mdx+Ndy$, $dV=Mdx+Ndy+Pdp$: the resulting formulæ of solution are, $N=0$, and $N-\frac{dP}{dx}=0$, so if we put

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \&c.,$$
$$\left[p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ r = \frac{dq}{dx},\ \&c.\right]$$

then the resulting formula will be

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c. = 0$$

which, by the principle stated in page 52, will belong to problems of all classes involving definite expressions. This last formula, remarkable for the simplicity of its law, supersedes Euler's table of the values of P*, given in his former memoir. In fact, all problems, in which integral expressions do not enter, are solved by it; and although Euler himself, and afterwards Lagrange, very materially simplified, and expedited its proof, yet, as a formula of solution, it still remains as a final result of all researches on this subject.

Problems are classed according to the number of properties which they involve†; and the number determines that of the ordinates that must be made to vary. But, problems might be distinguished also from each other, by the order of differentials which they involve; and then the order would determine the number of terms to be used in the formula $dV = Mdx + Ndy + Pdp$ + &c. thus, if the curve were required, in which $\int (ax - y^2)y\,dx$ is a maximum, we have $V = (ax - y^2)y$, $dV = ay.dx + (ax - 3y^2)\,dy$: consequently, two terms are sufficient, namely, Mdx, Ndy, and the formula of solution is

$$N = 0, \text{ or } ax - 3y^2 = 0.$$

This problem is, according to Euler's classification, of the first class, and might be said also to be of the first order.

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* See p. 41. † See p. 35.

If the *brachystochrone*, with no limitation, were required, since the time

$$=\int\frac{ds}{\text{vel}^{y}.}=\int\frac{ds}{\sqrt{y}}=\int\frac{\sqrt{(dx^2+dy^2)}}{\sqrt{y}}=\int dx\frac{\sqrt{(1+p^2)}}{\sqrt{y}},$$

by comparing it with $\int V dx$, we have $V=\frac{\sqrt{(1+p^2)}}{\sqrt{y}}$ *,

and thence $dV=-\frac{\sqrt{(1+p^2)}}{2.y^{\frac{3}{2}}}.dy+\frac{p}{\sqrt{y(1+p^2)}}dp$; so that, the formula $Mdx+Ndy+Pdp$, or three terms are sufficient, one of which $Mdx=0$.

This problem of the first class, might be said to be of the second order.

Again, the solid of least resistance is of the same class and same order with the preceding problem; for since $\int\frac{y.dy^3}{dx^2+dy^2}=$ a minimum $=\int V dx$, by comparison, we have, $V=\frac{yp^3}{1+p^2}$, and $dV=\frac{p^3}{1+p^2}.dy+\frac{3p^3+p^4}{(1+p^2)^2}.y\,dp$.

But, if a curve were required, in which $\int\frac{d^2y}{ds}$ should be a minimum, then since $Vdx=\frac{d^2y}{ds}=\frac{d^2y}{dx^2}.\frac{dx}{\sqrt{(1+p^2)}}$

$V=\frac{q}{\sqrt{(1+p^2)}}$, and $dV=-\frac{pq}{(1+p^2)^{\frac{3}{2}}}.dp+\frac{1}{\sqrt{(1+p^2)}}.dq$

so that this problem would require four terms, or this formula

$$dV=Mdx+Ndy+Pdp+Qdq;$$

(in which M and $N=0$), and therefore might be said to

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* See p. 51.

be of the third order, and, according to Euler's classification, would be of the first class, if it depended on the minimum property alone.

The class of problems then determines the number of ordinates that must be made to vary; but, on what might be called their order, would depend the number of ordinates to be taken account of, in the computation of the variations. For, as more terms of the series $Mdx+Ndy+$&c. enter, more ordinates are required to estimate the variations of the terms: for instance, let $dV=Mdx+Ndy+Pdp$: then if the problem is of the first class, the variation of one ordinate is sufficient. Let* $BL=y'$ be the ordinate that varies, and let the two adjacent ordinates be y, y'', then it is plain, from the subjoined table, that three ordinates y, y', y'' are sufficient for the computation of the variations in

$Mdx+Ndy+Pdp$, and in $M'dx+N'dy'+P'.dp'$,

$y=AC$	$p=\frac{dy}{dx}=\frac{EL}{dx}$	Variation in $p=\frac{\nu}{dx}$.
$y'=BL$	$p'=\frac{dy'}{dx}=\frac{LI}{dx}$	variation in $p'=-\frac{\nu}{dx}$.
$y''=KD$	$p''=\frac{dy''}{dx}$	

The introduction of other ordinates, such as y''', y^{iv}, &c. or $y_{,}$ $y_{,,}$ &c. on either side, is plainly unnecessary.

If, however, $dq=\frac{d^2p}{dx^2}=\frac{d^3y}{dx^3}$ enters into the form, or if

$$dV=Mdx+Ndy+Pdp+Qdq,$$

* See Fig. p. 52.

then three ordinates are not sufficient: an additional one $y_{\prime}$ must be taken account of: for the element $V_{\prime}dx$ of the area analogous to Vdx, and immediately preceding it, is $M_{\prime}dx + N_{\prime}dy_{\prime} + P_{\prime}dp_{\prime} + Q_{\prime}.dq_{\prime}$; now, the last term $Q_{\prime}dq_{\prime}$ of this formula, is affected by the increment v of the ordinate y'; for, since $q=\frac{dp}{dx}=\frac{p'-p}{dx}, q_{\prime}=\frac{p-p_{\prime}}{dx}$; $\therefore q_{\prime}$ is varied, if p is varied; and p is varied, if y' is. Hence we must introduce into the computation an additional ordinate $y_{\prime}$; but that is sufficient; for none of the terms representing $V''.dx$, $V'''.dx$, &c. will be affected by the variation of the ordinate y'.

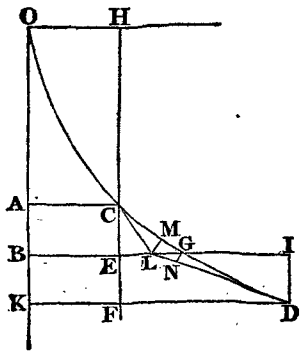
What has been said relates to those cases in which V involves no indefinite expressions; for if a term, such as Lds be introduced, a fourth ordinate must be taken account of. If the problem is of the second class, then two ordinates must be made to vary, and the number of ordinates that must be taken account of is easily determined from the preceding reasonings. We shall have occasion, however, to notice this case, when we speak of Euler's treatise on Isoperimetrical problems.

In the conclusion of his memoir, Euler notices, that his methods fail, or require peculiar artifices, if the quantity V in $\int Vdx$, contains the arc s, or other integral quantities. For, in that case, the principle of the whole curve possessing a certain property of maximum or minimum, if a portion of it does, is not true. For if V contains an integral expression, then the curve OCD may contain $\int Vdx$ a maximum or minimum,

although no separate portion of it has that property. If, indeed,

$$dV = Mdx + Ndy + P\,dp + \&c.$$

then, since the quantities x, y, $\frac{dy}{dx}$, or p depend on the



point C, the part CLD may be changed at pleasure, without any change being produced in OC; but if V contains an integral, then if CLD is changed, OC will take another value from $\int V dx$. In these cases then, the preceding methods of Euler, deduced from assuming a portion of the curve possessing the requisite property, fail: so likewise do the methods of Bernoulli and Taylor, which are founded on the same principle *.

There are cases, however, Euler observes, in which

* Bernoulli's has been already mentioned; and Taylor, in his Methodus Incrementorum, p. 67. prefixes to his solution, a similar one. Its truth depends, as it has been said, on the two portions adjacent to CLD not varying whilst CLD varies, which is to be granted, when $\int V dx$ contains no integral.

the ordinary solutions may be applied; although the quantity $\int V dx$ contains an integral: and these happen when the integral is by the conditions common, or the same to all curves, amongst which a required curve is to be found. Included in these cases, are the Isoperimetrical problems of the Bernoullis; for there s, the integral of $dx\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}$ is given: had it not been given, or if, instead of the Isoperimetrical property, another had been substituted, the solution of the second case proposed in James Bernoulli's *programma*, would have been faulty.

Very important objects were obtained by Euler in this memoir. The solution of problems involving differentials of any order; the invention of a formula including his former formulæ, which, to the number of twenty-four, he had inserted in a table; the partial solution of problems involving integral expressions; the establishment of his theorems and formulæ by easier processes *.

An author is usually, more than justly, fond of his last inventions: and Euler, by this memoir, thought he had nearly perfected the method of solving Isoperimetrical

* "Non solum faciliorem quam ante detexi viam ad solutiones hujusmodi problematum perveniendi, sed etiam omnes 24 formulas, quas ante tractaveram, in unicam sum complexus, &c." *Comm. Petrop.* tom. VIII. p. 159.

problems *: yet his methods were not without their imperfections. They afforded no general solutions of problems involving integral expressions; and, erroneous solutions, when the differential function depended on a quantity given solely by a differential equation not generally integrable: and the cause of these imperfections was the assumption of the principle, that the whole curve will be endowed with the property of maximum or minimum, if any portion whatever of it possess the same [see p. 52, 62.].

We are now led, both according to the order of historical and scientific succession, to the consideration of Euler's tract on Isoperimetrical problems.

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* "Triplici ratione illam priorem methodum ad majorem perfectionis gradum sum evecturus." *Comm. Petrop.* tom. VIII. p. 175.

CHAP. V.

Euler's Tract, entitled "Methodus inveniendi Lineas Curvas Proprietate Maximi Minimive gaudentes"— Distribution of Cases into absolute and relative Maxima and Minima—Rules for finding the Increment of Quantities dependent on their varied State — Formulæ of Solution.

THIS Work appeared in the year 1744, about three years after the publication of his last memoir *, on the same subject. It was intended, and, with a few exceptions, it must be conceded, to be a complete treatise; containing essentially all the requisite methods of solution, with great abundance and variety of examples and illustrations. There is wanting, however, to make it a perfect work, and on the subject, the best extant, a new algorithm; a more compendious process of establishing the theorems; and certain supplemental formulæ, that determine, not the nature of the curve, if a curve be the object of enquiry, but the conditions according to which it must be drawn. These desiderata were afterwards supplied by the fertile genius of Lagrange.

The former memoir contained, as it has been already stated, abundance of valuable matter, but ill arranged. The

* The volume of the Commentaries of Petersburg, in which this memoir is inserted, is said to be for the year 1736; which determines nothing concerning the dates in which the memoirs contained in it were written. Several of them, however, were undoubtedly written after 1736; since they contain accounts of observations made in 1740.

distribution and arrangement, however, of the present work is extremely luminous and regular. Absolute maxima and minima are first treated of, which concern curves that are to be determined solely by the property of maximum or minimum; such a curve is the *brachystochrone* *, which has the property of the least time, out of all curves whatever that can be drawn between two given points. The curve generating by its rotation round its axis the solid of least resistance, is another.

If $\int V dx$ be the analytical expression of the maximum or minimum, V may contain either determinate or indeterminate quantities, such as integrals. Euler first considers the former cases, that is, when V contains only quantities, such as x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. which are plainly determinate quantities, that is, of assignable value, when x or y is given.

After absolute, relative maxima and minima are treated of: these relate to curves that are to be determined not solely by the maximum property, but conjointly by that and other properties. Such a curve is the *brachystochrone* †, when the property of equal length becomes an additional condition; that is, when the curve of quickestdescent is required, not amongst all curves whatever, that can be drawn between two given points, but only amongst those that are of a given length: such also is the *brachystochrone*, when a third condition, that of equal area, is added ‡.

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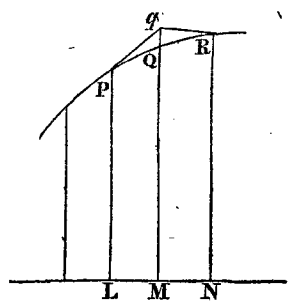
* See pp. 4, 9, 34. † See p. 18. ‡ See p. 47.

In these cases of relative maxima and minima, the quantity V, when $\int V dx$ represents a property, may, or may not, include integral expressions: and since, by an artifice like that which we have stated*, Euler reduces all questions in which are involved two or more properties, analytically expressed by $\int V dx$, $\int Y dx$, $\int X dx$, to this form,

$$\int V dx + a \int Y dx + b \int X dx + \&c.$$

the determination of all cases is reduced, ultimately, to that of an absolute maximum or minimum.

The method employed by Euler in treating the first and simplest cases of absolute maxima and minima, is similar to that used by him in the eighth volume of the Commentaries of Petersburg: thus, suppose the expression of the maximum property to be $\int V dx$, and V to be a determinate function† of x, y, $\frac{dy}{dx}$, &c.; then, as it has



appeared, the property belongs equally to the curve and to its element. Let PQR be a curve, and let its

* P. 56. † P. 62.

ordinates LP, MQ, NR, &c. be y', y'', y''', &c. and $Qq = v$; now $\int V dx$ is to be a minimum or maximum; suppose $\int V_{\cdot} dx$ to be its value up to the ordinate y, the one immediately preceding LP; then, beyond that ordinate, to the right of y, the value will be increased by $V dx + V' . dx + V'' . dx +$ &c.; so that, the whole value of $\int V dx$ will be

$$\int V_{\cdot} dx + (V + V' + V'' + \text{&c.})\, dx$$

a maximum or minimum, and in which, for the reasons just assigned,

$(V + V' + V'' + \text{&c.})\, dx$ must be a maximum or minimum, that is, $(dV + dV' + dV'' + \text{&c.})\, dx$ must be put equal to nothing *, when the changes arising in dV, dV', &c. from the translation of the point Q to q, or from the increment or *variation* v are properly expressed; or, if $dV = M dx + N dy + P dp + Q dq$, when the changes or variations of

$$dx\,(M dx + N dy + P dp + Q dq) \quad [1]$$
$$dx\,(M' dx + N' . dy' + P' . dp' + Q' . dq') \quad [2]$$
$$dx\,(M'' dx + N'' . dy'' + P'' dp'' + Q'' . dq'') \quad [3]$$

are expressed.

Now in the Figure p. 67. the ordinate y'' is changed by the quantity v, and since

$$p = \frac{y' - y}{dx}, \quad q = \frac{p' - p}{dx} = \frac{y'' - 2y' + y}{dx^2},$$
$$p' = \frac{y'' - y'}{dx}, \quad q' = \frac{p'' - p'}{dx} = \frac{y''' - 2y'' + y'}{dx^2}, \quad [a]$$
$$p'' = \frac{y''' - y''}{dx}, \quad q'' = \frac{p''' - p''}{dx} = \frac{y^{v} - 2y''' + y''}{dx^2},$$

* See p. 53.

it is easy to see the change, or increment, or variation produced in p, p', &c. q, q', &c. from the change v in y'', which is the sole ordinate that admits of any: thus, the variation *

in p is $= 0$; in p' is $\frac{+v}{dx}$; in p'' is $\frac{-v}{dx}$; in q is $\frac{v}{dx^2}$; in q' is $\frac{-2v}{dx}$; in q'' is $\frac{v}{dx^2}$.

Hence, substituting in the three expressions [1], [2], [3],

the variation in [1] is $\frac{Q.v}{dx^2}\,dx$;

. . . . . in [2] is $\left(P'.\frac{v}{dx} - \frac{Q'.2v}{dx^2}\right)dx$;

. . . . . in [3] is $\left(N''.v - P''.\frac{v}{dx} + Q''.\frac{v}{dx^2}\right)dx$.

Hence, the whole variation is

$$\left(N'' - \frac{P''-P'}{dx} + \frac{Q''-2Q'+Q}{dx^2}\right)dx.v = 0;$$

but, $\frac{P''-P'}{dx} = \frac{dP'}{dx}$ (the increment of P' being diminished in infinitum,)

also $\frac{Q''-2Q'+Q}{dx^2} =$

$$\frac{(Q+2dQ+d^2Q)-2(Q+dQ)+Q}{dx^2} = \frac{d^2Q}{dx^2}.$$

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* Variation, now used technically, is not in this Tract used so by Euler.

Hence, retaining, for the sake of homogeneity, quantities of the same order,

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0,$$

which formula will solve all questions of absolute maxima that do not involve integral expressions, or differential expressions, of a higher order than dq, or $\frac{d^2y}{dx^2}$; and by means of the principle and formula stated in p. 56, will solve all questions of relative maxima that neither involve integral, nor differential expressions of a higher order than $\frac{d^2y}{dx^2}$.

There are two points, in the preceding process, that require explanation. The increment or variation v is attributed not to the ordinate y or y', but to y''; and three values of dV, such as $Mdx+Ndy+Pdp+Qdq$, only were taken. With regard to the first point, the value of $\int Vdx$, that is, $\int V_{,}dx$, from the left hand up to the ordinate y, is supposed to be determined, and not to be affected by any change in that portion of the curve that lies to the right of y; now if y'' be changed, this happens, or $\int V_{,}dx$ is not affected: for, an element of it to the left of y and immediately preceding y, would be, analogously to the mode of expression used,

$$M_{,}dx + N_{,}.dy_{,} + P_{,}.dp_{,} + Q_{,}.dq_{,}$$

and according to the table [a] p. 68, we should have

$$p_{,} = \frac{y - y_{,}}{dx}, \quad q_{,} = \frac{y' - 2y + y_{,}}{dx^2}$$

neither of which quantities depend on y'', the ordinate that

is changed; and consequently, these quantities would suffer no change or variation, and therefore $\int V dx$ will not be affected. But, if to y' instead of y'' the increment v had been attributed, then q would have been changed, and its variation would have been $\frac{v}{dx^2}$.

This explanation includes, in fact, that of the second point: if a fourth horizontal series

$$M'''.dx + N'''.dy''' + P'''.dp''' + Q'''.dq'''$$

had been taken, then in each term, there would have been no variation from the variation v of the ordinate y''. For, according to the table [a]

$p''' = \frac{y^{IV} - y'''}{dx}$, $q''' = \frac{y^{V} - 2y^{IV} + y'''}{dx^2}$, which quantities are independent of y''.

If the quantity V involve differentials of a higher order than d^2y, the variation v must be attributed to an ordinate more remote from y than y''; and more horizontal rows like [1], [2], [3], p. 68, must be taken into the computation. We may easily learn from the preceding explanation, the exact process that must be instituted; for instance, suppose

$$[b] \quad dV = Mdx + Ndy + Pdp + Qdq + Rdr$$

$$\left[r = \frac{dq}{dx} = \frac{d^2p}{dx^2} = \frac{d^3y}{dx^3}\right]$$

then, forming r', r'', &c. as p', p'', q', q'', &c. were formed in table [a] p. 68, we have

$$r = \frac{q' - q}{dx} = \frac{y''' - 3y'' + 3y' - y}{dx^3},$$

$$r' = \frac{q'' - q'}{dx} = \frac{y^{iv} - 3y''' + 3y'' - y'}{dx^3},$$

$$r'' = \frac{q''' - q''}{dx} = \frac{y^{v} - 3y^{iv} + 3y''' - y''}{dx^3},$$

$$r''' = \frac{q^{iv} - q'''}{dx} = \frac{y^{vi} - 3y^{v} + 3y^{iv} - y'''}{dx^3};$$

now, if ν be attributed to y'', then the preceding value of r, or $r_{\backprime} = \frac{y'' - 3y' + 3y - y_{\backprime}}{dx^3}$ would be affected by the change in r; therefore, we must, instead of y'', make the next ordinate, or y''', to vary by the quantity ν, and then r, and consequently, $\int V_{\backprime} dx$ will not be affected by any change in the curve that lies to the right of y: again, four rows like to [b] p. 71, are sufficient; for, if we introduce a fifth, since y^{iv}, p^{iv}, q^{iv}, r^{iv} do not involve y''', no change would be introduced in it, or its variation, arising from ν, would be nothing.

If the several variations in the four values of dV be collected and reduced, after the manner given in p. 69, the result will be

$$\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3}\right) . dx . \nu$$

which, by the nature of the question, that is, by the property of maximum, must equal nothing.

The above mode of computing that peculiar increment of $\int V dx$, which arises from a change in the curve itself, and which Lagrange technically, and for distinc-

tion, called the variation, is sufficiently plain and direct. The rule too that Euler gives *, is only in different terms, the very rule now in use; for instance, to find the variation of $y' \surd(1+p^2)$

take, says he, the differential of this quantity as in the ordinary calculus; and then, instead of the differentials of y', p, substitute those peculiar differentials, that is, variations, that arise from a change in one of the ordinates (y'); accordingly, the common differential being

$$(dy') \times \surd(1+p^2) + \frac{y'p}{\surd(1+p^2)}(dp)$$

the peculiar differential, or variation, if $(dy') = v$ and $(dp) = \frac{v}{dx}$, will be

$$v \surd(1+p^2) + \frac{y'p}{\surd(1+p^2)} \cdot \frac{v}{dx}.$$

In the preceding formula, V contains definite quantities, such as x, y, dy, &c. but it may contain an integral expression, or be of this form $T\int Z dx$: in such case, Euler employs a method similar to his preceding one; but, the method becomes very complicated: for then corresponding to V would be $T\int Z dx$, and since the consecutive value of T would be T', and of $\int Z dx$, $\int Z dx + Z.dx$, the quantity corresponding

to V' would be $T'\int Z dx + T'.Z dx$,

to V'' . . . . . $T''\int Z dx + T''Z dx + T''.Z'dx$,

and then from these expressions, dV, dV', &c. must be expressed, and their values put down, such as arise from the variation v of any ordinate as $y^{(n)}$.

* *Methodus Inveniendi*, p. 33, &c.

It will be easily perceived from what is already said, that this process must be very tedious; and it is not necessary more fully to explain it, since the method, with regard to its principle, contains nothing either novel or abstruse, and as a method of computation has been superseded by another more regular and concise.

Since $s = \int \sqrt{(dx^2 + dy^2)} = \int dx \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$.

or since s is an integral, if

$$dV = Mdx + Ndy + Lds,$$

the formula for the peculiar differential or variation of $\int V dx$ would be given by the preceding method; which formula is to be adopted to the exclusion of those other given (dV having the same expression) by Euler in the eighth volume of the Petersburg Commentaries.

V has been supposed to contain an integral $\int Z dx$, and we may go on farther, and suppose Z also to contain an integral such as $\int X dx$, or V to contain a double integral, and so on: the methods to be pursued for such cases will be similar to the preceding: they will, however, from their complication shew how inconvenient Euler's process is; although, if we attend to its management by the author himself, we cannot help admiring the singular art and dexterity with which, from a perplexed mass of symbols, he at length extricates compact and simple formulæ of solution.

Euler in the sixth volume of the Commentaries of Petersburg reduced problems of relative maxima and

minima to a dependance on as many similar equations, as the properties proposed. For instance, if the curve required was to possess two properties, the equations would be of the form $R.bg - S.ci$*, in which $S = R + dR$. But, Euler gives there no satisfactory nor general proof, that $S = R + dR$; but contents himself with saying, that from the analytical process, such will appear to be the value of S; and if S should not be under the proper form, he then directs the computist so to reduce the equation $R.bg - S.ci$, that the coefficient analogous to S, shall equal $R + dR$.

In the present Work, however, he gives a much more satisfactory proof of this important principle: the nature of which will be easily understood from one or two simple examples. Let

$$dV = Mdx + Ndy + Pdp,$$

and let ν be the variation of the ordinate y'' and ω that of y'; ν and ω representing quantities such as bg and ci are in Bernoulli's and Euler's former method†: then since [see p.68.]

$$p = \frac{y' - y}{dx}, \quad p' = \frac{y'' - y'}{dx}, \quad p'' = \frac{y''' - y''}{dx}$$

the variation of $p = \frac{\omega}{dx}$, of $p' = \frac{\nu - \omega}{dx}$, of $p'' = -\frac{\nu}{dx}$

consequently, the variation of $dV.dx = \frac{P\omega}{dx}.dx$

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* P. 37. † Pp. 15, 36.

the variation of $dV'.dx = \left(N'\omega + P'.\frac{\nu - \omega}{dx}\right)dx$

. . . . . of $dV''.dx = \left(N''.\nu - P''.\frac{\nu}{dx}\right)dx.$

Hence, the whole variation is [see pp. 55, 68.]

$$\left(N' - \frac{P' - P}{dx}\right).dx.\omega + \left(N'' - \frac{P'' - P'}{dx}\right).dx.\nu.$$

Now, N'' is the consecutive value of N', or $= N' + dN'$ $P'' - P'$ is the consecutive value of $P' - P$, or $= (P' - P)$ $+ d(P' - P)$; consequently, if we make $N' - \frac{P' - P}{dx} =$ R, the coefficient of the second term, or $N'' - \frac{P'' - P'}{dx}$, will be $R + dR$. Hence, the form of the resulting equation, will be

$$R\omega + (R + dR)\,\nu.$$

Again, suppose

$$dV = Mdx + Ndy + Pdp + Qdq;$$

here, if the increments ν and ω are attributed to y'' and y', we must introduce an ordinate $y_{\backprime}$ preceding y; since $q_{\backprime} = \frac{y' - 2y + y_{\backprime}}{dx^2}$. Hence, referring to the values of p, q, given, p. 68, and operating as before;

variation of $p_{\backprime} = 0$, of $p = \frac{\omega}{dx}$, of $p' = \frac{\nu - \omega}{dx}$, of $p'' = \frac{-\nu}{dx}$

of $q_{\backprime} = \frac{\omega}{dx^2}$, of $q = \frac{\nu - 2\omega}{dx^2}$, of $q' = \frac{-2\nu + \omega}{dx^2}$, of $q'' = \frac{\nu}{dx^2}$

Hence the variat^{n}. of $dV.dx = Q.\frac{\omega}{dx^2}.dx$,

. . . . . of $dV.dx = \left(P\frac{\omega}{dx} + Q.\frac{\nu - 2\omega}{dx^2}\right).dx$

. . . . . of $dV'.dx = \left(N'\omega + P'.\frac{\nu - \omega}{dx} + Q'.\frac{\omega - 2\nu}{dx^2}\right)dx$,

. . . . . of $dV''.dx = \left(N''.\nu - P''\frac{\nu}{dx} + Q''.\frac{\nu}{dx^2}\right).dx$

or, collecting the terms affected with ω, and ν, the variation is

$\left(N' - \frac{P' - P}{dx} + \frac{Q - 2Q + Q'}{dx^2}\right)\omega.dx + \left(N'' - \frac{P'' - P'}{dx} + \frac{Q - 2Q' + Q''}{dx^2}\right)\nu.dx$; where it is plain, the coefficient of the term involving νdx, is consecutive to the coefficient of the term involving $\omega.dx$; or is $R + dR$, if the coefficient of the first term is R; hence, as before, the resulting formula is

$$R.\omega + (R + dR).\nu.$$

The truth of this result may be also easily perceived from the form which the variations of p, p', p'', and of q, q', q'', &c. necessarily assume from their expressions in terms of the consecutive ordinates, y, y', y'', &c.

We may here again notice the important result, which Euler drew from the above formula. Since $R\omega + (R + dR)\nu$ is derived from a property, of maximum or other, represented by $\int V dx$; if another property $\int Y dx$ be added, a similar equation, $Q\omega + (Q + dQ)\nu$ may be derived: and exterminating ν, the resulting equation is

$$R + aQ = 0;$$ that is,

$$\text{if } V = M dx + N dy + P dp + \&c.$$
$$\text{and } Y = \mu . dx + \nu . dy + \pi . dp + \&c.$$
$$\text{we shall have } R + aQ = \left(N - \frac{dP}{dx} + \&c.\right) + a\left(\nu - \frac{d\pi}{dx} + \&c.\right)$$
$$= (N + a\nu) - \frac{d(P + a\pi)}{dx} + \&c.;$$

that is, the same formula that would result, if the property proposed had been

$$\int V dx + a \int Y . dx.$$

By this, all questions of *relative* are reduced to those of *absolute* maxima and minima: for, similar reasonings and properties hold, when the curve sought instead of two, has three, four, &c. properties: and if such properties be expressed by

$$\int V dx, \int Y dx, \int W dx, \int U dx, \&c.$$

then we must solve the question as one of absolute maximum and minimum; and inquire, what the curve is that has the property expressed by

$$\int V dx + a \int Y dx + b \int W dx + c \int U dx.$$

Euler, besides the cases already mentioned, solves also those, in which V contains quantities, neither determinate such as $x, y, p,$ &c. nor integrals; but expressed solely under the forms of differential equations. What we have given however, is sufficient to explain and illustrate Euler's method. The results of that method are, for the practical solution of problems, under a most convenient form. On that head there is nothing to desire. Neither is there any want of perspicuity in the principle or in the conduct of his method. It is the length of the opera-

tions attendant on his method, the want of mechanism in his calculus, that are objectionable *. These inconveniences Lagrange removed: but, as in such cases it not unfrequently happens, whilst he rendered the process of calculation more expeditious, he deprived its principles of a considerable portion of their plainness and perspicuity.

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* Euler, we learn from the historical account prefixed to vol. X. of the *Novi Comm. Petrop.*, was sensible of the inconveniences of his method: "Interim tamen ipsa methodus, etiamsi totum negotium satis expedite conficiat, tamen ipsi non satis naturalis est visa, propterea quod vis solutionis tota in consideratione elementorum curvæ investigandæ erat posita, ista vero quæstio facile ita adornari possit, ut ex geometria penitus ad solam analysin puram revocetur, &c... Tametsi autem auctor de hoc diu multumque esset meditatus, atque amicis hoc desiderium aperuisset tamen gloria primæ inventionis acutissimo Geometræ Taurinensi Lagrange erat reservata, qui sola analysi usus eandem plane solutionem est adeptus, quam auctor ex considerationibus geometricis elicuerat." *Novi Comm. Petrop.* tom. X. p. 12.

CHAP. VI.

Lagrange's Memoir — Use of an appropriate Symbol to denote the *Variation* of a Quantity — Rules for finding the Variation — New Process of deducing Euler's Formulæ—Invention of new Formulæ.

In the preceding account of Euler's memoir, it cannot have escaped notice, that some ambiguity and much circumlocution took place in describing the process of finding that peculiar increment which depends on the quantity ν, by which the ordinate y or y' is increased.

These inconveniences Lagrange obviated by a simple invention; that of a symbol such as δ *; which, analogously to the symbol d of the differential calculus, was to be the means of representing either a quantity or an operation. Thus, the quantities ν and ω, by which y and y' are increased, are symbolically denoted by δy, $\delta y'$: and as d in $d(ay^n + by'^m)$ signifies, that the operation of taking the differential of $ay^n + by'^m$ is to be made; so, δ in $\delta(ay^n + by'^m)$, signifies that the peculiar differential of $ay^n + by'^m$ is to be taken, when instead of dy, dy'; ν and ω, or δy, $\delta y'$, are to be used.

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* "Mais avant tout je dois avertir que comme cette methode exige que les mêmes quantités varient de deux manières differentes, pour ne pas confondre ces variations j'ai introduit dans mes calculs une nouvelle characteristique δ." *Miscell. Taurin.* tom. II. p. 174.

With this change in the system of symbols, and with certain changes in the processes of establishing the fundamental formulæ, Lagrange called his method, a new method of determining the maxima and minima of indefinite integrals. But Euler, resuming the subject in the 10th volume of the New Commentaries of Petersburg, p. 54, called it by a name which still adheres to it, *The Calculus of Variations.*

The variation of $ay^n + by'^m$ is $nay^{n-1}.\delta y + mby'^{m-1}.\delta y'$. For if we recur to the problems solved by Bernoulli and Euler *, we shall find the difference of two contiguous states of a quantity at its maximum or minimum put equal to nothing; not indeed the whole difference, but the first term of the difference; for the terms involving $(bn)^2$, $(co)^2$, &c. † are neglected. Now, the difference in that example, is $\frac{1}{\sqrt{Pb}} - \frac{1}{\sqrt{(Pb+bn)}}$; Pb or y being increased by bn: if y had been increased by dy, the first term of $\frac{1}{\sqrt{Pb}} - \frac{1}{\sqrt{(Pb+dy)}}$ would have been the common differential; consequently the peculiar differential, that which is now to be called the *variation*, differs from the common differential, only inasmuch as bn or v or δy takes the place of dy. The process then for finding the coefficient of the term involving bn or δy, is

* Pp. 7, 33, &c. † P. 19.

precisely the same as that for finding the coefficient of the term involving dy; that is, for finding, what technically* is called the differential coefficient. The rules must consequently be the same: in other words, the rules of the differential calculus, become, with the alteration pointed out, those of the calculus of variations.

Hence $\delta(y^n) = ny^{n-1}.\delta y$, which is the first term of $[(y+\delta y)^n - y^n]$ expanded.

Again, $\delta(ay^2+by^4)^{\frac{1}{4}} = \frac{1}{4}(ay^2+by^4)^{-\frac{3}{4}}.(2ay+4by^3)\delta y$.

If δx be the variation of x; then since the differential of xy, or $d(xy)$ is $= x dy + y dx$,
the variation of xy, or $\delta(xy)$ is $= x\delta y + y\delta x$.

If we take the example, p. 73,

since $d[y.\sqrt{(1+p^2)}] = dy\sqrt{(1+p^2)} + \frac{yp}{\sqrt{(1+p^2)}}dp$

$$\delta[y\sqrt{(1+p^2)}] = \delta y\sqrt{(1+p^2)} + \frac{yp}{\sqrt{(1+p^2)}}.\delta p$$

In p. 68. $p = \frac{y'-y}{dx} = \frac{dy}{dx}$;

$$\therefore \delta p = \frac{\delta y' - \delta y}{dx} \text{ or } = \frac{\delta dy}{dx}, \text{ but } \frac{\delta y' - \delta y}{dx} = \frac{d\delta y}{dx}.$$

Hence, $\delta p = \frac{\delta dy}{dx}$ or $= \frac{d\delta y}{dx}$.

If $V = ax^3\sqrt{(1+y^2)}$ hyp. log. $p \times e^{mq}$
[e = number whose hyp. log. = 1,]

* *Princ. Anal. Calc.* p. 74.

then since $dV = 3ax^2\sqrt{(1+y^2)}\,\text{h.l.}\,p.e^{mq}.dx + \frac{ax^3y\,\text{hl}\,p.e^{mq}}{\sqrt{(1+y^2)}}dy$

$+ax^3\sqrt{(1+y^2)}\,\frac{1}{p}\,e^{mq}.dp + ax^3\sqrt{(1+y^2)}\,\text{h.l.}\,p.e^{mq}m.dq$

therefore putting M, N, P, Q, for the coefficients of the terms involving dx, dy, dp, dq, we have $\delta V = M.\delta x + N.\delta y + P.\delta p + Q.\delta q$, or expressing M, N, P, Q, as partial differential coefficients *,

$$\delta V = \frac{dV}{dx}\delta x + \frac{dV}{dy}\delta y + \frac{dV}{dp}\delta p + \frac{dV}{dq}\delta q\,\dagger.$$

Hence, generally, whatever be the function V,

if $dV = Mdx + Ndy + Pdp + Qdq + \&c.$

then $\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + \&c.$

Since the processes for finding the differential and variation differ only in the symbols dy, δy, which are arbitrary; it is plain, if both operations are to be performed on an analytical expression, that it is matter of

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* *Princ. Anal. Calc.* p. 79.

† This rule is not only in the first solutions of Isoperimetrical problems, virtually acted upon, but expressed. " Il faut bien remarquer que la difference des fonctions de deux lignes comme RO, RT $(y, y+\delta y)$ qui se surpassent d'une quantité TO infinitement petite du second genre, se trouve en *differentiant simplement* la fonction de RO, et en multipliant par TO (δy) *ce* qui en vient, ayant omis les differentielles. Par exemple, si RL (Y) fonction de RO (y) etoit seulement la puissance n de la même RO (y), en quoi consiste le cas de mon frère, c'est a dire, que si la courbe BH etoit une parabole du degré n, alors LM (δY) ou $RO^n - RT^n$ $[(y+\delta y)^n - y^n]$ seroit $= n.RO^{n-1} \times TO\,(ny^{n-1}.\delta y)$." John Bernoulli. *Acad. des Sciences*, 1706, p. 236; also *Opera*, tom. I. p. 424; and Euler, *Methodus Inveniendi*, p. 33, &c. gives the very rule and method for finding the *variation*, which Lagrange invested with appropriate symbols, [see p. 73.]

indifference, which operation is performed first: or, if the symbols d, δ, meet together denoting operations, we may, at our pleasure, change their order: for instance $d\delta y$ and δdy are alike significant; for $d\delta y$ means the first term of two successive values of δy, or $= \delta\,(y+dy)-\delta y = \delta y + \delta dy - \delta y = \delta dy$: again, if V for instance be a function of y; then

$$dV = \frac{dV}{dy}.dy, \quad \text{and } \delta dV = \frac{d^2V}{dy^2}.\delta y.dy$$

$$\delta V = \frac{dV}{dy}.\delta y, \quad \text{and } d\delta V = \frac{d^2V}{dy^2}.dy.\delta y;$$

$$\therefore \delta dV = d\delta V,$$

or, in a particular instance, when $V = y^n$,

$$d\,(y^n) = 1^{st} \text{ term of } \left[(y+dy)^n - y^n\right] = ny^{n-1}.dy$$

$$\delta d(y^n) = 1^{st} \text{ term of } ndy \times \left[(y+\delta y)^{n-1} - y^{n-1}\right] = n\,(n-1)y^{n-2}.dy.\delta y,$$

$$\delta(y^n) = 1^{st} \text{ term of } \left[(y+\delta y)^n - y^n\right] = ny^{n-1}.\delta y$$

$$d\delta\,(y^n) = 1^{st} \text{ term of } n\delta y \times \left[(y+dy)^{n-1} - y^{n-1}\right] = n(n-1)y^{n-2}.\delta y.dy;$$

$$\therefore \delta d\,(y^n) = d\delta\,(y^n).$$

And, by similar processes, $d^2\delta V = \delta d^2 V = \delta d d V = d\delta dV$

$$d^3\delta V = \delta d^3 V = d\delta d^2 V = d^2\delta dV.$$

This rule is, in Lagrange's method, of the greatest importance; it is an essential part of it. Amongst other uses, it enables us when an integral is concerned, to introduce the symbol δ within the symbol $(\int)$ of the integral: thus, since the symbols d and $\int$ indicate reverse operations,

$V = d\int V$; $\therefore \delta V = \delta d\int V =$ [by Rule, p. 84.] $d\delta\int V$.
Hence, taking the integrals on each side

$$\int \delta V = \int d\delta \int V = \delta \int V \quad . \quad . \quad . \quad [a]$$

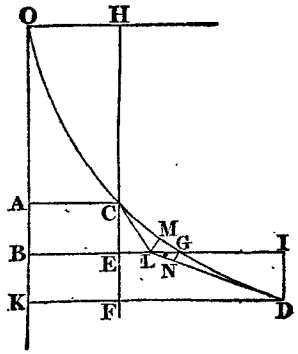
This result may be easily extended to double and treble integrals: for if $V = \int W$, then $\delta V = \delta \int W = \int \delta W$ by $[a]$;

$\therefore \int \delta V = \int\int \delta W$: but $\int \delta V = \delta \int V = \delta \int\int W$, consequently

$$\delta \int\int W = \int\int \delta W.$$

Before we proceed to explain the last improvement made by Lagrange, we will give, under the symbols of his new Algorithm, another solution of the *brachystochrone*, [see pp. 4, 9.]

Let $AC = y$, $BL = y'$, $CL = ds$, $LD = ds'$, $OA = x$, $OB = x'$, $g = 32\frac{1}{2}$ feet; then the velocity $= \sqrt{2gx}$,



hence, $\dfrac{ds}{\sqrt{(2gx)}} + \dfrac{ds'}{\sqrt{(2gx')}} =$ a minimum;

$$\therefore \delta\left(\frac{ds}{\sqrt{x}} + \frac{ds'}{\sqrt{x'}}\right) = 0, \text{ or } \frac{d\delta s}{\sqrt{x}} + \frac{d\delta s'}{\sqrt{x'}} = 0,$$

but $ds = \sqrt{(dx^2 + dy^2)}$; $\therefore d\delta s = \dfrac{dy.d\delta y}{ds}$ and $d\delta s' =$

$\frac{dy'.d\delta y'}{ds'}$; hence, $\frac{dy.d\delta y}{ds\sqrt{x}}+\frac{dy'.d\delta y'}{ds'\sqrt{x'}}=0$ [a]. But

$$KD=AC+dy+dy';$$

$\therefore$ since KD and AC are not changed in the translation of L to G, $dy+dy'$ is constant; $\therefore d\delta y=-d\delta y'$, and substituting in [$a$], l. l. $\frac{dy}{ds\sqrt{x}}=\frac{dy'}{ds'\sqrt{x'}}=$ a constant quantity, the property of the cycloid.

This solution is on the same principle, as the first and second, but is less peculiar and geometrical than either; instead of the similar triangles in the second, one of the processes of the new method above described, has been used.

The rules in pp. 84, 85, relate to the second improvement made by Lagrange in the calculus of variations which we shall now describe. If we refer to the method which Euler used for deducing the formula, $N-\frac{dP}{dx}+$&c., we shall perceive that its length and complication arises from the integral $\int V dx$ being broken down into parcels and elements Vdx, $V'dx$, &c. and from the calculation of the variations in the differential expressions that represent dV, dV', dV'', &c. [pp. 68, 69.]

Lagrange precluded the necessity of this resolution of $\int V dx$ into its elements, by combining with the preceding *variation* processes, an integral process. This will be understood from the solution of the following problem, in which it is required to find an expression for $\delta\int V dx$, V being a function of x, y, p, q, r, &c. the law of

the formation of p, q, r, &c. being $p=\frac{dy}{dx}$, $q=\frac{dp}{dx}$, $r=\frac{dq}{dx}$, &c.

$\delta \int V dx = \int \delta (V dx)$ [p. 85. l. 3.] $= \int \delta V . dx + \int V d\delta x$.

Now on the principle of this formula, $\int x dy = xy - \int y dx$

$$\int V d\delta x = V . \delta x - \int dV . \delta x.$$

Hence, $\delta \int V dx = V . \delta x + \int (\delta V . dx - dV . \delta x)$

Now since $\delta V = M\delta x + N\delta y + P\delta p + Q\delta q +$ &c.

if $dV = Mdx + Ndy + Pdp + Qdq +$ &c.

$$\delta V . dx - dV . \delta x = N(\delta y . dx - dy . \delta x) + P(\delta p . dx - dp . \delta x) + Q(\delta q dx - dq . \delta x) + \text{\&c.}$$

$$= Ndx(\delta y - p . \delta x) + Pdx(\delta p - q\delta x) + Qdx(\delta q - r\delta x) + \text{\&c.}$$

$\left(\text{since } p=\frac{dy}{dx},\ q=\frac{dp}{dx},\ \text{\&c.}\right)$.

Now $p=\frac{dy}{dx}$; $\therefore \delta p = \frac{d\delta y}{dx} - \frac{dy . d\delta x}{dx^2} = \frac{1}{dx}(d\delta y - p . d\delta x)$;

$$\therefore \delta p - q\delta x = \frac{1}{dx}(d\delta y - p d\delta x - q\delta x . dx) =$$

$$\frac{1}{dx}(d\delta y - p d\delta x - dp . \delta x) = \frac{1}{dx} . d(\delta y - p\delta x)$$

Now since the quantities p, q, r, &c. are formed after the same law, the same relation that subsists between $\delta p - q\delta x$ and $\delta y - p\delta x$, must subsist between $\delta q - r\delta x$ and $\delta p - q\delta x$, and so on; but the relation between the two former quantities is thus expressed:

$$\delta p - q\delta x = \frac{1}{dx} d(\delta y - p\delta x)$$

$$\therefore \delta q - r\delta x = \frac{1}{dx} d(\delta p - q\delta x) = * \frac{1}{dx^2} . d^2 (\delta y - p\delta x)$$

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* dx being constant, otherwise $= \frac{1}{dx} d\frac{1}{dx} . d(\delta y - p\delta x)$.

and $\delta r - s\delta x = \frac{1}{dx} d(\delta q - r\delta x) = \frac{1}{dx^2} . d^2 (\delta y - p\delta x)$.

Hence, for the purpose of abridgment, putting $\delta y - p\delta x = \delta\omega$, we have,

$$\delta \int V dx = V.\delta x + \int dx \left(N.\delta\omega + \frac{P d\delta\omega}{dx} + \frac{Q d^2\delta\omega}{dx^2} + \&c.\right).$$

But by Formula, p. 87. l. 3.

$$\int P d\delta\omega = P.\delta\omega - \int dP.\delta\omega,$$

$$\int Q d^2\delta\omega = Q d\delta\omega - \int dQ d\delta\omega = Q d\delta\omega - dQ.\delta\omega + \int d^2Q.\delta\omega$$

$$\int R d^2\delta\omega = R d\delta\omega - \int dR d\delta\omega$$

$$= R d^2\delta\omega - dR d\delta\omega + \int d^2R.d\delta\omega,$$

$$= R.d^2\delta\omega - dR.d\delta\omega + d^2R\delta\omega - \int d^3R . \delta\omega.$$

Hence, collecting quantities involving like symbols,

$$\delta \int V dx = V\delta x + \int dx.\delta\omega \left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c.\right) *$$

$$[A] \qquad + \left(P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \&c.\right).\delta\omega$$

$$+ \left(Q - \frac{dR}{dx} + \&c.\right) d\delta\omega$$

$$+ \&c. \qquad + K.$$

The quantity K represents the sum of the corrections introduced by the integrations.

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* $\frac{dP}{dx}$, $\frac{d^2\mathfrak{Q}}{dx^2}$, &c. are not partial differential coefficients, but the first, second, &c. entire differentials of P, $\mathfrak{Q}$, &c. divided by dx, dx^2, &c. respectively. They ought in strictness, to be written $\frac{1}{dx}.dP$, $\frac{1}{dx^2}.d^2\mathfrak{Q}$, &c.

If x be supposed to have no *variation*, or if $\delta x = 0$, then $\delta\omega = \delta y - p\delta x = \delta y$; and consequently we shall have the variation of $\delta \int V dx$ by omitting in the preceding form $V\delta x$, and writing δy, $d\delta y$, &c. instead of $\delta\omega$, $d\delta\omega$, &c.

The preceding value of the variation may be supposed to be taken between two limits corresponding to values a, and b, of x. Let $y_\circ$, $P_\circ$, $Q_\circ$, &c. be the values of y, P, Q, &c. at the first limit when $x = a$; and and y_I, P_I, Q_I, &c. when $x = b$;

$$\text{then, } \delta \int V dx = V_\text{I}.\delta x_\text{I} - V_\circ.\delta x_\circ + \int dx\, \delta\omega \left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.\right) + \left(P_\text{I} - \frac{dQ_\text{I}}{dx} + \&c.\right) \delta\omega_\text{I},$$

$$[Aa] \quad \ldots\ldots - \left(P_\circ - \frac{dQ_\circ}{dx} + \&c.\right) \delta\omega_\circ, + \&c.$$

If at the two limits $x = a$, $x = b$, the values of $y_\circ$, y_I are given, then $\delta\omega_\text{I}$, $\delta\omega_\circ$ are both equal to nothing, (since $\delta\omega = \delta y - p\delta x$); and in this case the variation of $\int V dx$ is reduced to the quantity under the integral sign ($\int$), which, in the case of $\int V dx =$ a maximum, since $\delta \int V dx = 0$, must equal nothing; or

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c. = 0.$$

This last formula, is that which Euler, by the method

described in the last Chapter, arrived at*; and it is here deduced on his hypothesis of the evanescence of δy_0, δy_1, &c.

The formula $N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} -$ &c. is not, however, equal 0, solely in the case when δx_0, δx_1, δy_0, δy_1, are equal 0, but also, when these are, from certain equations, assignable quantities. For, the formula [*Aa*] p. 89, is composed of two parts: one, affected by the integral sign, expresses the sum of all the separate variations throughout the whole extent of the curve or integrated quantity; the other part, independant of the integral sign, is affected only by the variations at the extreme points, and therefore cannot by any combination with the other, (which by changing δx and δy may be varied at will) form a sum equal to nothing. Hence, since $\delta \int V dx$ must $= 0$; each part separately, the one under the integral sign $\int$, the other not affected by it, must $= 0$.

M. Lagrange is the inventor of that part of the

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* *Methodus Inveniendi*, &c. Prob. V. Lagrange, *Miscell. Taurin.* tom. II., says, "Mais les formules de cet auteur (Euler) sont moins generales que les notres 1° &c.—2°, parcequ'il suppose que le premier et le dernier point de la courbe sont fixés, &c." Euler also, in his memoir subsequent to Lagrange's [*Novi. Comm.* tom X. p. 110.] acknowledges that his first formula did not contain the absolute or definite parts involving $\delta\omega$, $d\delta\omega$, &c. "Neque tamen hæ partes absolutæ frustra sunt inventæ, sed singularem præbent usum, ad quem methodus mea prior, quæ tantum æquationem $N - \frac{dP}{dx} +$ &c. $= 0$ suppeditavit, minus est accommodata; quam ob causam hæc methodus illi longe est anteferenda."

general formula which is not affected by the integral sign. Its use, as it will be shewn hereafter, is great and extensive: without it, the solution of problems would be incomplete; for, Euler's formula, that under the integral sign, merely determines in general terms the relation of x and y; undoubtedly the chief, but not the sole object of search.

We will now proceed to that, which, in fact, is Lagrange's second problem*.

It is required to find an expression for $\int V dx$, in which $dV = T dt$, dt not being similar to dy, dp, dq, &c., but determined by this equation $t = \int Z dx$.

By p. 87, $\delta \int V dx = V\delta x + \int (\delta V . dx - dV . \delta x) \ldots . [1]$

but $\delta V = T . \delta t$; $\therefore \delta V dx - dV . \delta x = T dx . \delta t - T \delta x . dt$.

again, by [1] $\delta t = \delta \int Z dx = Z . \delta x + \int (\delta Z . dx - dZ . \delta x)$;

$\therefore T dx . \delta t - T \delta x . dt = TZ . dx . \delta x - TZ . dx . \delta x +$

$T dx \int (\delta Z dx - dZ \delta x)$,

$= T dx \int (\delta Z dx - dZ . \delta x)$

and $\therefore \delta \int V dx = V \delta x + \int T dx \int (\delta Z dx - dZ \delta x)$.

Now by virtue of this formula $\int (v du) = vu - \int (u dv)$, if h be put $= \int T dx$, the latter part of the value of $\delta \int V dx$ is, $h \int (\delta Z dx - dZ \delta x) - \int h (\delta Z dx - dZ \delta x)$

Also by Form [A], p. 88. if $dZ = M dx + N dy + P dp +$ &c.

$$h \int (\delta Z dx - dZ \delta x) = h \int dx . \delta \omega \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \text{&c.} \right) +$$

$h . L \ldots . [2]$

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* *Miscellanea Taurinensia*, tom. II, p. 183.

and $\int h(\delta Z\,dx - dZ\delta x) =$

$$\int dx.\delta\omega\left(N - \frac{d(hP)}{dx} + \frac{d^2(hQ)}{dx^2} - \&c.\right) + hL' \quad \ldots \quad [3]$$

(L, L' being put for the definite parts,)

Subtracting this from [2] and adding $V\delta x$, we shall have the value of $\delta\int V dx$.

In general, it is required to find the whole variation of $\int V dx$ from $x=0$, for instance, to $x=a$. In such case, let the whole integral of $T dx = H$; then the part [2], of the

$$\text{variation} = H\int dx\,\delta\omega\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.\right) + HL$$

$$= \int dx\,\delta\omega\left(HN - \frac{d(HP)}{dx} + \frac{d^2(HQ)}{dx^2} - \&c.\right) + HL$$

For since H is constant, $H\int N = \int HN$, and $H.dP = d(HP)$; subtracting therefore, as before, [3] from [2], we have

$$\delta\int V dx = V\delta x$$

$$+\int dx.\delta\omega\left((H-h)N - \frac{d(H-h)P}{dx} + \frac{d^2(H-h)Q}{dx^2} - \&c.\right)$$

$$[B] \ldots\ldots + \left((H-h)P - \frac{d(H-h)Q}{dx} + \&c.\right)\delta\omega$$

$$+ \&c.$$

If we put $H-h$, that is, $H - \int T dx = k$, and suppose δx, $\delta\omega$, the variations at the limits, to equal nothing, we shall have

$$\delta\int V dx = \int dx.\delta\omega\left(kN - \frac{d(kP)}{dx} + \frac{d^2(kQ)}{dx^2} - \&c.\right)$$

which is Euler's formula, *Method. Inveniendi*, &c. p. 91, and the subject of his third Chapter.

If $dV=\mu.dx+\nu.dy+\pi.dp+\&c.+Tdt$,

which is Lagrange's case. *Misc. Taur.* tom. II. p. 183.

$$\delta\int Vdx=V\delta x+\int dx.\delta\omega\left(\nu-\frac{d\pi}{dx}+\&c.\right)\ldots[C]$$
$$+\int dx\delta\omega\left(kN-\frac{d(kP)}{dx}+\&c.\right)$$
$$+S,$$

S being put for the sum of the definite terms.

If $dt=ds=\sqrt{(dx^2+dy^2)}$ and $dV=\mu dx+\nu dy+Tds$

$$\delta\int Vdx=\int dx\delta\omega\left[\nu-\frac{d\left([H-\int Tdx]\frac{p}{\sqrt{(1+p^2)}}\right)}{dx}\right]..[C\tau]$$

not taking account of the definite parts;

For, $dt=ds=dx\sqrt{(1+p^2)}=Zdx$; $\therefore Z=\sqrt{(1+p^2)}$ and $dZ=\frac{p}{\sqrt{(1+p^2)}}.dp$; which, compared with the value of dZ [p. 91,] gives $N=0$, $P=\frac{p}{\sqrt{(1+p^2)}}$, $Q=0$, &c.

Hence, in the case of maximum or minimum, since $\delta\int Vdx=0$, there results this equation of solution,

$$\nu.dx-d\left([H-\int Tdx]\frac{p}{\sqrt{(1+p^2)}}\right)=0;$$

whereas Euler, *Novi Comm.* tom. VI. p. 141, gives this

$$\nu+T\frac{p}{\sqrt{(1+p^2)}}=0,$$

v answering to M, T to L, and $\frac{p}{\sqrt{(1+p^2)}}$ being $= q = \frac{dy}{ds}$.

This is the case before alluded to [p. 61.]; when it was remarked that Euler's formulæ contained in the sixth and eighth volume of the Petersburg Commentaries, were erroneous, when V contained an integral; which integral in the above instance is s.

In the third problem of Lagrange [p. 185.], it is required to find the variation of $\int V$ when V is given simply by a differential equation involving no differentials of V higher than the first.

Let $dV - X.dx + Udx = 0$ be the differential equation, X being a function of x, y, p, q, &c. and U a function of x, y, p, q, &c. and of V; then taking the differential, and supposing dx to be constant,

$$d^2V - \left(\frac{dX}{dx}dx + \frac{dX}{dy}dy + \frac{dX}{dp}dp + \&c.\right)dx + \left(\frac{dU}{dx}.dx + \frac{dU}{dy}dy + \frac{dU}{dp}dp + \&c.\right)dx + \frac{dU}{dV}.dV.dx = 0,$$

or $d^2V - d\phi.dx + \frac{dU}{dV}.dV.dx = 0$, ... [a]

Substituting $d\phi$ for the collection of terms involving dx, dy, dp, &c. Hence, if instead of deducing the differential in the last operation, we suppose the variation to be deduced,

$$d\delta V - \delta\phi.dx + T.\delta V.dx = 0 \qquad \left[T = \frac{dU}{dV}\right]$$

multiply by λ, and $\lambda\, d\delta V - (\lambda\delta\varphi)\, dx + (\lambda\, T.\delta V).dx = 0$,

or, $d(\lambda\delta V) - d\lambda.\delta V - (\lambda\delta\varphi)\, dx + (\lambda T.\delta V).dx = 0 \ldots [b]$

Assume the sum of the second and fourth terms to $=0$;

$\therefore d\lambda - \lambda T.dx = 0$; and $\frac{d\lambda}{\lambda} = T dx$, and integrating $\lambda = e^{\int T dx}$

[e = number whose hyp. log. = 1.]

Hence substituting this value of λ in the sum of the first and third terms,

$$e^{\int T dx}.\delta V = \int e^{\int T dx}.\delta\varphi.dx;$$

$$\therefore \int \delta V \text{ or } \delta \int V = \int e^{-\int T dx}.\int e^{\int T dx}.\delta\varphi.dx.$$

Euler, by a different process, solves this problem in the third Chapter of his treatise on Isoperimetricals.

Euler, however, stopped at this problem; and did not, as Lagrange has done, proceed on to those which besides dV, involve d^2V, d^3V, &c. Suppose U to be a function of x, y, p, &c., V and of $\frac{dV}{dx}$; then in taking the differential [a] p. 94. of the equation $dV - Xdx + Udx = 0$, an additional term $\frac{dU}{dv}.d^2V\left(v = \frac{dV}{dx}\right)$ will be introduced, and consequently in the variation, this term $\frac{dU}{dv}.d\delta V$; therefore when the equation is multiplied by λ (for the same process must be used,) there will be an additional term $\lambda T'.d\delta V \left(T' = \frac{dU}{dv}\right) = d(\lambda T'.\delta V) - d(\lambda T') \times \delta V$;

$\therefore$ equation [b] l. 2, would be

$d(\lambda\delta V) - d\lambda.\delta V - \lambda\delta\varphi.dx + (\lambda T.\delta V)\,dx + d(\lambda T'.\delta V) - d(\lambda T')\,\delta V.$

Hence, making the sum of the 2^{d}, 4th, and 6th terms = 0,

$$-d\lambda + \lambda.T dx - d(\lambda T') = 0 \ldots [c]$$

and $d[(\lambda + \lambda T')\delta V] - \lambda\delta\varphi\, dx = 0 \ldots . [d]$

in which two equations, λ must be deduced from the first and substituted in the second.

The process is the same if U besides a function of the former quantities, is also a function of d^2V; for then in the equation of variation there would be introduced a term $T''.d^2\delta V$ $\left(T'' = \frac{dV}{d\omega},\ \omega = \frac{d\nu}{dx}\right)$; and consequently in the equation multiplied by λ, a term $= \lambda T''.d^2\delta V =$ $d(\lambda T'' d\delta V) - d(\lambda T'').d\delta V = d(\lambda T''.d\delta V) - d^2(\lambda T''.\delta V)$ $+ d^2\lambda T'' \times \delta V.$

Hence the equation [c] l. 4. will become

$$-d\lambda + \lambda.T dx - d(\lambda T') + d^2.(\lambda T'') = 0,$$

and the equation [d] l. 5.

$d[(\lambda + \lambda T' - d(\lambda T''))\,\delta V] - d(\lambda T''.d\delta V) - \lambda\delta\varphi\, dx = 0$: and similarly, if differentials of a still higher order are introduced.

In the preceding cases, the maxima and minima depend on a function of one variable quantity alone: but a variable quantity z may be introduced, a function of two others x and y. For instance, suppose it were required to find amongst equal solids, that which is bounded by the least surface, (which is the problem given by

Lagrange in his first Appendix, p. 188.): then, if the points of the surface be referred to three rectangular co-ordinates x, y, z, we have

$$dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) . dy$$

$\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$, being the partial differential coefficients *, we have also the solidity $= \int\int z . dx . dy$, and the surface $=$

$$\int\int dx . dy \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]}$$

the integrals being taken, first relatively to x, then relatively to y. The problem therefore analytically expressed is

$$[a] \ldots\ldots \int\int \delta (z . dx . dy)$$

$$+ a \int\int \delta . (dx . dy) \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} = 0,$$

$$\text{now } \delta \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} =$$

$$\left[\frac{dz}{dx} \delta \left(\frac{dz}{dx}\right) + \frac{dz}{dy} \delta \left(\frac{dz}{dy}\right)\right] . \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{-\frac{1}{2}}$$

$$\text{make } \frac{dz}{dx} . \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{-\frac{1}{2}} = P,$$

$$\frac{dz}{dy} \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{-\frac{1}{2}} = Q,$$

$$\text{then } \int P \delta \left(\frac{dz}{dx}\right) = \int P \frac{d \delta z}{dx} = P . \frac{\delta z}{dx} - \int \frac{dP}{dx} \delta z;$$

$$\text{similarly } \int Q \delta \left(\frac{dz}{dy}\right) \ldots\ldots = Q \frac{\delta z}{dy} - \int \frac{dQ}{dy} \delta z.$$

* *Princ. Anal. Calc.* p. 79.

Hence the equation [a] p. 97. l. 10. becomes

$$\int\int dx.dy.\delta z + a\int P dy.\delta z + a\int Q dx.\delta z$$

$$- a\int\int dx.dy \frac{dP}{dx}\delta z - a\int\int dx.dy \frac{dQ}{dy}\delta z = 0.$$

Hence, making the sum of the second and third terms, which are affected with one sign of integration, and belong to the extreme points of the curve surface, $=0$

$P dy + Q dx = 0$, and consequently the remaining terms $=0$, or, $1 - a.\frac{dP}{dx} - a.\frac{dQ}{dy} = 0$, [$p$]

now the condition under which

$$M dx + N dy$$

is a complete differential, is this *

$$\frac{dM}{dy} = \frac{dN}{dx}$$

Hence, putting $N = x - aP$, $M = aQ$, we have

$$a\frac{dQ}{dy} = 1 - a.\frac{dP}{dx}, \text{ or, } 1 - a\frac{dP}{dx} - a\frac{dQ}{dy} = 0,$$

which is the equation [p] l. 8.; hence, $(x - aP)\,dy + aQdx$ is a complete differen^{l}.: but that, $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$ may be real, $\frac{dz}{dx}dx + \frac{dz}{dy}dy$ must be also a complete differential; that is, $\frac{d^2z}{dx.dy}$ must $= \frac{d^2z}{dy.dx}$. These two

* Woodhouse, *Princ. Anal. Calc.* p. 88 : also Euler, *Calc. Int.* p. 315.

conditions therefore must be fulfilled, in order that the problem may be solved.

If r be the radius of a sphere, and α, β, γ, the three co-ordinates of its center,

$$r = \sqrt{[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]};$$

$$\therefore dz = \frac{1}{(z-\gamma)} \cdot [(x-\alpha)\,dx + (y-\beta)\,dy];$$

$$\therefore \frac{dz}{dx} = \frac{1}{(z-\gamma)}\,(x-\alpha), \text{ and } \frac{dz}{dy} = \frac{1}{(z-\gamma)}(y-\beta);$$

$\therefore$ (putting for $z-\gamma$ its value) $\frac{d^2z}{dx.dy} = \frac{d^2z}{dy.dx}$ one condition therefore is satisfied:

again $P = \frac{x-\alpha}{r}$, $Q = \frac{y-\beta}{r}$; $\therefore (x - aP)\,dy + aQdx =$

$\left(1 - \frac{a}{r}\right) xdy + \frac{a}{r} y\,dx + \frac{a\alpha}{r} dy - \frac{a\beta}{r}.dx$: which is a complete differential, making $1 - \frac{a}{r} = \frac{a}{r}$; $\therefore$ the portion of the sphere corresponding to x, y, z, satisfies the problem.

The formulæ of solution, on which the nature of the curve, or the relation of x and y, depends, were all, excepting the two last, invented by Euler. To Lagrange belongs the merit of having deduced them by neater processes. The latter author, however, is the sole inventor of those definite and absolute formulæ [see p. 88. l. 13, 14.] which are requisite for the complete solution of Isoperimetrical problems.

In his first memoir, Lagrange seems not exactly to have comprehended the nature of these definite formulæ.

He drew some conclusions, which the Chevalier Borda * proved to be not general. These defects, however, M. Lagrange remedied in a subsequent memoir in the fourth volume of the Miscellanea Taurinensia: and without acknowledging the detection of the defects, or rather, with a faint endeavour of denying it, extended his formulæ, more accurately applied them, but confirmed the truth of M. Borda's results.

In the same memoir Lagrange considers the subject under a new point of view, and gives a method of solution including all his former ones.

The substance of Lagrange's first researches on this subject have been given, with some deviation, not essential, from their mode. Euler's last manner † of treating the subject has been followed. That is commended and adopted by Lagrange in the latest of his publications ‡. In the same Tract he has resumed the consideration of the general problem, and deduced formulæ applicable to all cases.

This last method of Lagrange's is distinguished rather by its mode of treating the question, than by any thing novel in its principles; and therefore, it will be separately considered in the following Chapter, which the Student who hastens towards the end of the inquiry, may, without inconvenience, pass over.

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* *Acad. des Sciences*, 1767. † *Novi Comm. Petrop.* tom. X. p. 51.
‡ *Leçons sur le Calcul des Fonctions.*

CHAP. VII.

Lagrange's general Method of treating Isoperimetrical Problems—Equation of Limits—Cases of relative Maxima and Minima reduced to those of Absolute.

LET V be a function of x, y, p, q, z, p', q', &c.

$$p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ p' = \frac{dz}{dx},\ q' = \frac{dp'}{dx},$$

and let $dV = Mdx + Ndy + Pdp + Qdq + \&c.$
$$+ \nu.dz + \pi.dp' + \sigma.dq' + \&c.$$

then, if $Y = N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.$

$$Y' = P - \frac{dQ}{dx} + \&c.$$

$$Y'' = Q - \&c.$$

$$\&c. = \&c.$$

$$Z = \nu - \frac{d\pi}{dx} + \frac{d^2\sigma}{dx^2} - \&c.$$

$$Z' = \pi - \frac{d\sigma}{dx} + \&c.$$

$$Z'' = \sigma - \&c.$$

$$\&c. = \&c.$$

we shall have, by processes already described, [pp. 87, 88.]

$$\delta \int V dx = V\delta x + \int dx.\delta\omega.Y + \int dx.\delta\omega' Z$$
$$+ Y'.\delta\omega + Z'.\delta\omega'$$
$$+ Y''.d\delta\omega + Z''.d\delta\omega'$$
$$+ \&c.$$

Hence, in the case of a maximum, when $\delta \int V dx = 0$,

$$Y.\delta\omega + Z.\delta\omega' = 0,$$

or $Y\delta y + Z\delta z = 0$, if $\delta x = 0$, since [p. 88.]

$\delta\omega = \delta y - p\delta x$, and $\delta\omega' = \delta z - p'.\delta x$

If y, z, are quantities independent of each other, then, $Y = 0$, and $Z = 0$.

But, if the quantities y and z are connected together by an equation, such as $f(x, y, z) = 0$, or $W = 0$, then, supposing x invariable, we have

$$\frac{dW}{dy}.\delta y + \frac{dW}{dz}.\delta z = 0,$$

which combined with the former equation [l. 3.] gives

$$Y.\frac{dW}{dz} - Z.\frac{dW}{dy} = 0.$$

The same result will be obtained, if x be supposed to vary; for then

$$\frac{dW}{dx}\delta x + \frac{dW}{dy}\delta y + \frac{dW}{dz}\delta z = 0,$$

which combined with the common differential equation

$$\frac{dW}{dx}.dx + \frac{dW}{dy}.dy + \frac{dW}{dz}.dz = 0,$$

gives, $\frac{dW}{dy}.\left(\delta y - \frac{dy}{dx}\delta x\right) + \frac{dW}{dz}\left(\delta z - \frac{dz}{dx}\delta x\right) = 0$;

or, $\frac{dW}{dy}.\delta\omega + \frac{dW}{dz}.\delta\omega' = 0$,

which combined with $Y\delta\omega + Z\delta\omega'$, gives

$$Y.\frac{dW}{dz} - Z.\frac{dW}{dy} = 0, \text{ as before.}$$

Hence, this conclusion follows; that the variation of x in no wise affects the general equation of maximum or minimum, but solely the equation at the limits.

The equation at the limits depends on the parts of the formula for $\delta \int V dx$, which are definite and freed from the integral sign, and is

$$\begin{aligned} &= V_1.\delta x_1 - V_0.\delta x_0 \\ &\quad + Y'_1.\delta \omega_1 + Z'_1.\delta \omega'_1 + \&c. \\ &\quad - Y'_0.\delta \omega_0 - Z'_0.\delta \omega'_0 + \&c. \end{aligned}$$

V_1, V_0, δx_1, δx_0, &c. representing the values of V, δx, &c. at the end and beginning of the integral.

Instead of eliminating $\delta \omega$, $\delta \omega'$, by the combination of the two equations [p. 102. l. 19, 20.] we may multiply the latter by an indeterminate quantity λ, add it to the first equation, and then determine the resulting equation, by the elimination of λ; thus

$$Y.\delta \omega + Z.\delta \omega' + \lambda \frac{dW}{dy}.\delta \omega + \lambda \frac{dW}{dz} \delta \omega' = 0.$$

Eliminate λ from the two equations

$$Y + \lambda \frac{dW}{dy} = 0, \text{ and } Z + \lambda \frac{dW}{dz} = 0,$$

and there results, as before,

$$Y.\frac{dW}{dz} - Z.\frac{dW}{dy} = 0,$$

which combined with $W = 0$, will give the values of y and z in terms of x.

This last method extended, will afford a general formula of solution; for, suppose W to be a function, not of x, y, z, only, but also of the differentials of these quantities; that is, suppose

$$dW = M'.dx + N'.dy + P'.dp + Q'.dq + \&c.$$
$$+ \nu'.dz + \pi'.dp' + \sigma'.dq' + \&c.$$

and let $(Y) = \lambda N' - \frac{d(\lambda P')}{dx} + \frac{d^2(\lambda Q')}{dx^2} - \&c.$

$$(Y') = \lambda P' - \frac{d(\lambda Q')}{dx} + \&c.$$

&c.

$$(Z) = \lambda\nu' - \frac{d(\lambda\pi')}{dx} + \&c.$$

$$(Z') = \lambda\pi' - \frac{d(\lambda\sigma')}{dx} + \&c.$$

&c.

then we shall have

$$\delta\int V dx + \delta\int(\lambda W).dx = (V + \lambda W).\delta x$$
$$+ \int[Y + (Y).dx.\delta\omega] + \int[Z + (Z)dx\,\delta\omega']$$
$$+ [Y' + (Y')]\delta\omega + [Z' + (Z')]\delta\omega'$$
$$+ \&c.$$

In this formula since $W = 0$, $\lambda W.\delta x = 0$, and since $\delta\omega$, $\delta\omega'$ are independent quantities, we have, on principles such as have been already stated,

$$Y + (Y) = 0, \qquad Z + (Z) = 0,$$

whence λ is to be eliminated; and then by the aid of the equation of condition $W = 0$, y and z are to be determined in terms of x.

This method comprehends all the former ones which

were separately instituted [pp. 88, 92.] to determine the variation of $\int V dx$.

1st. When dV contains merely x, y, p, q, &c. and no relation is assigned between x, y, z, then the equation $W = 0$ is not introduced, and the formula becomes $Y = 0$, $Z = 0$;

$$\text{or } N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. = 0,$$

$$\text{and } \nu - \frac{d\pi}{dx} + \frac{d^2\sigma}{dx^2} - \&c. = 0.$$

2dly, If V contains an integral expression; that is, if

$$dV = M dx + N dy + P dp + \&c. + T dt,$$

in which $t = \int S dx$, then the equation of condition $W = 0$ becomes $dt - S.dx = 0$, and consequently $\int \left(\frac{dt}{dx} - S\right) dx = a$. Hence, substituting t for z, we have

$$Y = N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.$$

$$Z = T \qquad [\text{since } \nu = T,\ \sigma = 0,\ \&c.]$$

$$-(Y) = \lambda N' - \frac{d(\lambda P')}{dx} + \frac{d^2(\lambda Q')}{dx^2} - \&c.$$

$$[\text{if } dS = M'.dx + N'.dy + \&c.]$$

$$(Z) = -\frac{d\lambda}{dx} \qquad [\text{since } \nu' = 0,\ \pi' = 1,\ \sigma' = 0,\ \&c.]$$

hence we have from the equation $Z + (Z) = 0$.

$$T - \frac{d\lambda}{dx} = 0, \text{ whence, } \lambda = \int T dx = \int T dx - H,$$

if $\int T dx = H$, when $\lambda = 0$: substituting this value of λ in the equation $Y + (Y) = 0$, there results

$$N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-\&c.+(H-\int T dx)N'-\frac{d[(H-\int T dx)P]}{dx}$$

+&c. as before, p. 92.

3dly, If $dV-\phi.dx+U.dx=0$, and U is a function of V, this answers to the equation $W=0$, and $W=\frac{dV}{dx}-\phi+U$, and $dW=T.dV+d.\frac{dV}{dx}-d\phi$; instead of z, consider V to be the variable quantity, then in

$dW=d\phi+v'.dz+\pi'.dp'+$ &c. $v'=T$, $\pi'=1$,

the other coefficients being 0;

Hence, $Y=0$, $(Y)=\lambda N'-\frac{d(\lambda P')}{dx}+$ &c.

$$Z=0,\ (Z)=\lambda T-\frac{d\lambda}{dx};$$

$\therefore Z+(Z)=0$, or $\lambda T-\frac{d\lambda}{dx}=0$, whence $\lambda=e^{\int T dx}$ as before, p. 95.

The same method applies to equations in which higher differentials than the first are involved; for instance, to equations such as

$$t''\frac{d^3V}{dx^3}+t'.\frac{d^2V}{dx^2}+t.\frac{dV}{dx}+\phi=0.$$

In the variation of $\int V dx$, the parts under the integral sign, that is, $Y+(Y)$, and $Z+(Z)$ equal nothing; there remains therefore, supposing $\delta x=0$,

$$\delta\int V dx=[Y'+(Y')]\delta y+[Z'+(Z')]\delta z+[Y''+(Y'')]d\delta y+[Z''+(Z'')]d\delta z+\&c.$$

which expression must be substituted in the equation of limits, that is, in

$$\delta U_1 - \delta U_0 = 0,$$

putting $U = \int V dx$.

At the limits, there may be particular relations between x and y; for instance, it may be required to draw the brachystochrone between certain curves. Let $L = 0$ be the equation to the curve at the first point, and $M = 0$ at the last point, that is, let L be a function of x_0, y_0, and M a function of x_1, y_1; then

$$\frac{dL}{dx}.\delta x_0 + \frac{dL}{dy}.\delta y_0 = 0, \text{ and } \frac{dM}{dx}\delta x_1 + \frac{dM}{dy}\delta y_1 = 0;$$

and since, by taking the differential equations, we have

$$\frac{dL}{dx}.dx_0 + \frac{dL}{dy}.dy_0 = 0, \text{ and } \frac{dM}{dx}dx_1 + \frac{dM}{dy}dy_1 = 0,$$

by elimination, there results

$$\frac{dy_0}{dx_0}.\delta x_0 - \delta y_0 = 0, \text{ and } \frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0,$$

from these equations, and from the equation of limits $\delta U_1 - \delta U_0 = 0$, which must contain δx_1, δx_0, δy_1, δy_0, the values of δx_0, δy_0 must be eliminated, and the resulting equation will determine the conditions of the problem that must be fulfilled at the limits.

The same conclusion will result, if, instead of an elimination, we multiply the equations $\delta L = 0$, $\delta M = 0$ by two indeterminate quantities λ, μ, and then add them to the equation of limits, which will become

$$\delta U_1 - \delta U_0 + \lambda \delta L + \mu \delta M = 0.$$

This equation will contain terms affected with δx_0, δx_1, δy_0, δy_1, the coefficients of which are separately to be made equal nothing.

The origin of the abscissas and ordinates (when the discussion is concerning curves) is supposed to remain constant; but, it may be supposed to vary, which will be equivalent to the hypothesis of making quantities, such as a, b, contained in V, to vary. In that case, if

$$\delta V = M\delta x + N\delta y + P\delta p + \&c. + A\delta a + B\delta b + \&c.$$

we have, as $M = \frac{dV}{dx}$, and $N = \frac{dV}{dy}$ &c. $A = \frac{dV}{da}$, and $B = \frac{dV}{db}$; and consequently the variation $\delta \int V dx$ will =

$$\int \delta y . dx \left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. \right)$$
$$+ \int \delta a dx . A + \int \delta b dx . B$$
$$+ \delta y \left(P - \frac{dQ}{dx} + \&c. \right)$$
$$+ \&c.$$

now the equation to the curve depends on the equation $N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.$ and the additional terms $\delta a \int A dx$, $\delta b \int B dx$, or $\delta a \int \frac{dV}{da} dx$, $\delta b \int \frac{dV}{db} dx$, will affect the equation of limits.

If $x = x' - a$, then $\delta x = \delta x' - \delta a$; therefore, if x' be the same, $\delta x = -\delta a$; consequently the term $A\delta a$, introduced by reason of the variation of the origin of the co-ordinates, will be affected with a negative sign.

The methods described in this Chapter comprehend all cases belonging to absolute maxima and minima: they extend also, if we employ Euler's reasoning*, to relative maxima and minima; or independently of the

* See p. 78.

last author's method, which may be thought not perfectly satisfactory, they may be extended by the following process, which is Lagrange's.

Let u be the function, the integral of which is, to have, within the assigned limits, a determinate value. Let s be its integral, then $ds - u dx = 0$; consider this to be an equation of condition, such as $L = 0$; then, since $\lambda \delta L$ was added [p. 107.] to the variation of $\int V dx$, we must now add

$$\lambda \delta (ds - u dx) \ldots . \quad [a]$$

the first of this, $\lambda \delta ds = \lambda d \delta s = d(\lambda \delta s) - d\lambda . \delta s$; a term therefore $- d\lambda . \delta s$ will be introduced under the integral sign, which, since δs is an arbitrary quantity, must $= 0$; $\therefore d\lambda = 0$, or $\lambda = a$, a constant quantity. The term $d(\lambda \delta s)$ will solely affect the equation of limits, and since the whole variation must be taken between s_1 and s_0, $a \delta s_1 - a \delta s_0$ must be added to the equation of limits, which, by the hypothesis of s having a determinate value, must $= 0$.

Hence there remains of $[a]$ l. 10, only $- a \delta (u dx)$; consequently, the whole variation is reduced to this $\delta \int (V - au) dx$: whence the rule before given, p. 78, which reduces relative maxima and minima to absolute, is derived; for the formula directs us to find the conditions of the absolute maximum or minimum of $\int (V - au) dx$, a being a constant quantity.

We now proceed to the last Chapter of this Work, in which, the formulæ previously established, will, with some slight alteration, be applied to the solution of problems.

CHAP. VIII.

Particular Formulæ deduced from the General one, for the Purpose of facilitating the Solution of Problems—Problems solved.

EULER in his treatise has deduced from his general formula, several subordinate ones, limited indeed, but materially expediting the solution of problems. These will be first described.

In the general formula,

$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \&c \ldots [A]$

suppose M, Q, and all coefficients excepting N, P, to equal 0; then, $dV = Ndy + Pdp$; but generally [p. 89.]

$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. = 0$ [a]; in this case, therefore,

$N - \frac{dP}{dx} = 0$; consequently, $Ndy - dP.\frac{dy}{dx} = 0$; or, since

$p = \frac{dy}{dx}$, $Ndy - p.dP = 0$; and substituting, in l. 10,

$dV = p.dP + Pdp = d(Pp)$; whence, by integration,

$V = Pp + c \ldots\ldots$ [b], c being the correction.

If M is not $= 0$, then $V = \int Mdx + Pp + c \ldots$ [c].

Let $M = 0$, $N = 0$, and all the coefficients after Q; then $dV = Pdp + Qdq$, but the general formula [a] l. 11. in this case becomes

$$-\frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0;$$

whence, $P = \int \frac{d^2Q}{dx^2} dx + c = \frac{dQ}{dx} + c$;

multiply this by dp, and since $\frac{dp}{dx} = q$, we have

$$P.dp = q.dQ + c.dp$$

hence, substituting, in the equation, [*A*] p. 110.

$$dV = q.dQ + Qdq + c.dp = d(Qq) + c.dp$$

and integrating,

$$V = Qq + cp + c' \ldots\ldots [d]$$

c, c', being the corrections.

If M does not $= 0$, we must add, to the above form, the term $\int M dx$.

If $M = 0$, but N is not $= 0$, that is, if the form be

$$dV = Ndy + Pdp + Qdq$$

then, since $N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0$,

we have, multiplying by dy, which $= pdx$,

$$Ndy - p.dP + p.\frac{d^2Q}{dx^2}.dx = 0.$$

But, $p.\frac{d^2Q}{dx^2}dx = d\left(p\frac{dQ}{dx}\right) - dp.\frac{dQ}{dx}$

$$= d\left(p.\frac{dQ}{dx}\right) - q.\frac{dQ}{dx}.dx.$$

Hence, substituting, in l. 12.

$$dV = pdP + Pdp - d\left(p.\frac{dQ}{dx}\right) + qdQ + Qdq;$$

consequently, $V = Pp + Qq - p.\frac{dQ}{dx} + c \ldots [e]$.

If $dV = Pdp + Rdr$,

then the formula [*a*] p. 110. is reduced to

$$-\frac{dP}{dx}-\frac{d^3R}{dx^3}=0;$$

consequently, $P=-\frac{d^2R}{dx^2}+c$, and $Pdp=c.dp-dp.\frac{d^2R}{dx^2}$;

hence, since $dp=q.dx$, we have, by substitution,

$$dV=c.dp+Rdr-q.\frac{d^2R}{dx^2}.dx$$

$$=c.dp+Rdr-d\left(q\frac{dR}{dx}\right)+dq.\frac{dR}{dx}$$

$$[\text{since } dq=r\,dx]=c.dp+Rdr+r\,dR-d\left(q.\frac{dR}{dx}\right)$$

Hence integrating, and adding the correction c',

$$V=cp+Rr-q.\frac{dR}{dx}+c' \ldots . \quad [f]$$

These forms are sufficient for the solution of the following problems.

Prob. 1.

Required the relation of x and y, such that $\int(ax-y^2).ydx$ shall be a maximum or minimum.

Comparing this expression with $\int V dx$, $V=axy-y^3$; consequently, $dV=ay.dx+(ax-3y^2)\,dy$;

which, compared with the general formula [*A*] p. 110. that is, with $dV=Mdx+Ndy+Pdp+Qdq+$&c.

gives $M=ay$, $N=ax-3y^2$, $P=0$, $Q=0$;

consequently, since $N-\frac{dP}{dx}+$&c. $=0$, in this case

$$ax-3y^2=0, \text{ and } y=\sqrt{\frac{ax}{3}}.$$

PROB. 2.

Required the shortest curve that can be drawn between two points, or between two curves.

Here $\int ds$ or $\int dx\sqrt{(1+p^2)}$ = a minimum; $\therefore V=\sqrt{(1+p^2)}$ and $dV=\frac{pdp}{\sqrt{(1+p^2)}}$; $M, N,$ &c. $=0$: and $P=\frac{p}{\sqrt{(1+p^2)}}$; consequently, the formula [a] p. 110, becomes $-d\left(\frac{p}{\sqrt{1+p^2}}\right)=0$; $\therefore p=a\sqrt{(1+p^2)}$ and p or $\frac{dy}{dx}=\frac{a}{\sqrt{(1-a^2)}}$, whence, by integration, $y\sqrt{(1-a^2)}=ax+c$, an equation to a right line.

PROB. 3.

Required the curve of quickest descent between two given points.

The time $=\int\frac{ds}{\sqrt{y}}=\int\frac{\sqrt{(dx^2+dy^2)}}{\sqrt{y}}=\int\frac{\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}}{\sqrt{y}}.dx$

$=\int\frac{\sqrt{(1+p^2)}}{\sqrt{y}}.dx$, which compared with $\int V dx$ gives $V=\frac{\sqrt{(1+p^2)}}{\sqrt{y}}$; whence $dV=-\frac{\sqrt{(1+p^2)}}{2y^{\frac{3}{2}}}dy+\frac{p}{\sqrt{y}\sqrt{(1+p^2)}}dp$, which compared with Form [A] p. 110, gives $M=0$, $N=-\frac{\sqrt{(1+p^2)}}{2y^{\frac{3}{2}}}$, $P=\frac{p}{\sqrt{[y(1+p^2)]}}$, $Q=0$. Now by the Form [b] p. 110, $V=Pp+c$; $\therefore$ in this case

$$\frac{\sqrt{(1+p^2)}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}.\sqrt{(1+p^2)}} + c,$$

whence $\frac{1}{\sqrt{y}\sqrt{(1+p^2)}} = c$, or $\frac{dx}{\sqrt{y}\sqrt{(dx^2+dy^2)}} = c$

and finally, $dx = \frac{c\sqrt{y}}{\sqrt{(1-c^2y)}}.dy$, an equation to a cycloid. See p. 5.

PROB. 4.

Required the relation of x and y when $\int(x^2+y^2)^n.ds =$ a maximum [see Euler's *Methodus Inveniendi*, &c. p. 52.]

Here $V=(x^2+y^2)^n \times \sqrt{(1+p^2)}$, since $ds = dx\sqrt{(1+p^2)}$ and, $dV =$

$$(2nx\,dx+2ny\,dy)(x^2+y^2)^{n-1}\sqrt{(1+p^2)} + \frac{(x^2+y^2)^n.p}{\sqrt{(1+p^2)}}dp;$$

$\therefore M = 2nx\,(x^2+y^2)^{n-1}\sqrt{(1+p^2)}$, $N = 2ny\,(x^2+y^2)^{n-1}\sqrt{(1+p^2)}$, and $P = \frac{(x^2+y^2)^n.p}{\sqrt{(1+p^2)}}$.

Here it is more convenient to employ the general formula,

$$N - \frac{dP}{dx} + \&c. = 0,$$

which, since

$$dP = \frac{(x^2+y^2)^{n-1}}{\sqrt{(1+p^2)}}.\left(2npx\,dx + 2npy\,dy + \frac{(x^2+y^2)\,dp}{1+p^2}\right)$$

by reduction becomes

$$\frac{2n(y\,dx - x\,dy)}{x^2+y^2} = \frac{dp}{1+p^2} \ldots\ldots [h],$$

if $\theta =$ arc, whose tangent $= \frac{x}{y}$, then $d\theta = \frac{y\,dx - x\,dy}{x^2+y^2}$;

consequently, integrating the equation [*h*], p. 114, $2n\theta = A + c$, A being an arc, whose tangent $= p$, or $\frac{dy}{dx}$

$$\therefore \frac{x}{y} = \text{tan.}\,\theta = \text{tan.}\,\frac{1}{2n}(A + c).$$

PROB. 5.

Required the curve which by a revolution round its axis generates the solid of least resistance. [see Newton, p. 324, 3d ed. Euler's *Methodus Inveniendi*, &c. p. 51. Simpson's *Fluxions*, p. 487, Ed. 1750. Emerson's *Fluxions*, p. 183, 3d ed. Lacroix's, *Calcul. Diff.* tom. II. p. 698.]

The resistance $= \int \frac{y.dy^3}{dx^2 + dy^2}$, which, compared with $\int V dx$, gives

$$V = \frac{yp^3}{1+p^2} \quad \left(\text{since } p = \frac{dy}{dx}\right), \text{ whence}$$

$$dV = \frac{p^3}{1+p^2}.dy + \frac{3yp^2 + yp^4}{(1+p^2)^2}.dp;$$

consequently, $M = 0$, $N = \frac{p^3}{1+p^2}$, $P = \frac{3yp^2 + yp^4}{(1+p^2)^2}$.

Hence by the Form [*b*] p. 110, using $-c$ for the correct^n,

$$\frac{yp^3}{1+p^2} = \frac{3yp^3 + yp^5}{(1+p^2)^2} - c$$

whence, by reduction,

$$c.(1+p^2)^2 = 2yp^3, \text{ and } y = \frac{c}{2p^3} + \frac{c}{p} + \frac{cp}{2};$$

but, $pdx = dy = -\frac{3c}{2p^4}dp - \frac{c}{p^2}dp + \frac{c}{2}dp$;

$$\therefore dx = -\frac{3c}{2p^5}dp - \frac{c}{p^3}dp + \frac{c}{2}.\frac{dp}{p}$$

and $x = \frac{3c}{8p^4} + \frac{c}{2p^2} + \frac{c}{2}.\text{hyp. log.}\, p + c'$

which equation must be combined with $c(1+p^2)^2 = 2yp^3$.

PROB. 6.

Required the curve in which $\int yxds$ is a maximum, [see Euler, *Methodus Inveniendi*, &c. p. 52.]

Since, $yxds = yx\sqrt{(1+p^2)}.dx$, $V = yx\sqrt{(1+p^2)}$ and $dV = y\sqrt{(1+p^2)}dx + x\sqrt{(1+p^2)}\,dy + \frac{xyp}{\sqrt{(1+p^2)}}.dp$;

$$\therefore M = y\sqrt{(1+p^2)},\ N = x\sqrt{(1+p^2)},\ P = \frac{xyp}{\sqrt{(1+p^2)}}$$

In this case therefore, we must use the Form [c] p. 110, and then $yx\sqrt{(1+p^2)} = \int y\sqrt{(1+p^2)}.dx + \frac{xyp^2}{\sqrt{(1+p^2)}} + c$

$$\text{or, } \frac{yx}{\sqrt{(1+p^2)}} = c + \int y\sqrt{(1+p^2)}.dx;$$

which by reduction leads to a differential equation of the second order.

PROB. 7.

Required the curve, or in other terms, the relation of x and y when $\int\frac{d^2y}{ds} =$ a maximum. (See Euler, *Comm. Acad. Petrop.* tom. VIII. p. 171.)

$$\frac{d^2y}{ds} = \frac{d^2y}{dx\sqrt{(1+p^2)}} = \frac{d^2y}{dx^2}.\frac{dx}{\sqrt{(1+p^2)}};\ \therefore V = \frac{q}{\sqrt{(1+p^2)}}$$

and $dV = -\frac{qp}{(1+p^2)^{\frac{3}{2}}}.dp + \frac{1}{\sqrt{(1+p^2)}}.dq$;

$\therefore M = 0,\ N = 0,\ P = -\frac{qp}{(1+p^2)^{\frac{3}{2}}},\ Q = \frac{1}{\sqrt{(1+p^2)}}$;

$\therefore$ by the Form [*d*] p. 111,

$$\frac{q}{\sqrt{(1+p^2)}} = \frac{q}{\sqrt{(1+p^2)}} + cp + c',$$

whence, p or $\frac{dy}{dx} = -\frac{c'}{c}$, and $cy = c'' - c'x$; an equation to a right line.

PROB. 8.

Required the curve in which $\int \frac{d^2y}{y.ds}$ = a maximum.

[see *Comm. Acad. Petrop.* tom. VIII. p. 171.]

$$\frac{d^2y}{y.ds} = \frac{d^2y}{dx^2.y\sqrt{(1+p^2)}}.dx;\ \therefore V = \frac{q}{y\sqrt{(1+p^2)}}$$

and $dV = -\frac{q}{y^2\sqrt{(1+p^2)}}.dy - \frac{qp}{y(1+p^2)^{\frac{3}{2}}}.dp + \frac{1}{y\sqrt{(1+p^2)}}dq$;

$\therefore M = 0,\ N = -\frac{q}{y^2\sqrt{(1+p^2)}},\ P = -\frac{qp}{y.(1+p^2)^{\frac{3}{2}}},$

$Q = \frac{1}{y\sqrt{(1+p^2)}}$; $\therefore$ by the Form [*e*] p. 111,

$$\left[\text{since } dQ = -\frac{dy}{y^2\sqrt{(1+p^2)}} - \frac{p\,dp}{y(1+p^2)^{\frac{3}{2}}}\right]$$

$$\frac{q}{y\sqrt{(1+p^2)}} = -\frac{qp^2}{y(1+p^2)^{\frac{3}{2}}} + \frac{q}{y\sqrt{(1+p^2)}} + \frac{p^2}{y^2\sqrt{(1+p^2)}} + \frac{qp^2}{y(1+p^2)^{\frac{3}{2}}} - c,$$

consequently, $\frac{p^2}{y^2\sqrt{(1+p^2)}} = c$, and

$$p^2 = \frac{c^2y^4}{2} \pm cy^2\sqrt{\left(1 + \frac{c^2y^4}{4}\right)}$$

and since $p = \frac{dy}{dx}$, $dx = \frac{\sqrt{2}.dy}{y\sqrt{c^2y^2 \pm c\sqrt{(4+c^2y^4)}}}$.

Prob. 9.

Required the curve which, within its own arc, its evolute, and radius of curvature, shall contain the least area*. [see Euler, *Comm. Acad. Petrop.* tom. VIII. p. 169; also *Methodus Inveniendi*, &c. p. 64.]

The radius of curvature $= \frac{ds^3}{-dx.d^2y}$ † $= \frac{(dx^2+dy^2)^{\frac{3}{2}}}{-dx.d^2y}$

$= -\frac{(1+p^2)^{\frac{3}{2}}}{dx.d^2y}dx^3 = -\frac{(1+p^2)^{\frac{3}{2}}}{q}$, since $q = \frac{dp}{dx} = \frac{d^2y}{dx^2}$.

Hence, the differential of the area $= \frac{\text{radius} \times ds}{2} =$

$-\frac{(1+p^2)^{\frac{3}{2}}}{q}\sqrt{(1+p^2)}dx = -\frac{(1+p^2)^2}{q}.dx$, which compared with Vdx, gives $V = -\frac{(1+p^2)^2}{q}$; consequently,

$$dV = -\frac{4p}{q}(1+p^2).dp + \frac{(1+p^2)^2}{q^2}dq;$$

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* The area is AOR in Simpson's *Diagram*, p. 78; *Fluxions*, ed. 1750.

† Simpson, p. 72. Vince, p. 149, First Edit. Woodhouse *Anal. Calc.* p. 180.

$\therefore M = 0,\ N = 0,\ P = -\frac{4p}{q}(1+p^2),\ Q = \frac{(1+p^2)^2}{q^2}$.

$\therefore$ by Form $[d]$ p. 111, that is, $V = Qq + cp + c'$, we have

$$-\frac{(1+p^2)^2}{q} = \frac{(1+p^2)^2}{q} + cp + c',$$

or $-\frac{2(1+p^2)^2}{q} = cp + c'$; but $q = \frac{dp}{dx}$,

$$\therefore -2(1+p^2)^{\frac{3}{2}}\sqrt{(1+p^2)}.dx = cp\,dp + c'.dp;$$

or, since, $ds = \sqrt{(1+p^2)}.dx$

$$-2\,ds = \frac{cp\,dp}{(1+p^2)^{\frac{3}{2}}} + \frac{c'.dp}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore -2s = -\frac{c}{\sqrt{(1+p^2)}} + \frac{c'.p}{\sqrt{(1+p^2)}} + c''$$

$$= \frac{c'p - c}{\sqrt{(1+p^2)}} + c''.$$

If $c,\ c'' = 0,\ -2s = \frac{c'p}{\sqrt{(1+p^2)}}$, which shews the curve to be a cycloid: for by p. 5. $\frac{dy}{dx}$ or $p = \sqrt{\left(\frac{a-y}{y}\right)}$

$\therefore \frac{p}{\sqrt{(1+p^2)}} = \sqrt{\frac{(a-y)}{a}}$, and the arc of a cycloid measured from its vertex $= 2$ chord of generating circle $= 2\sqrt{a.(a-y)} = \frac{2ap}{\sqrt{(1+p^2)}}$.

Prob. 10.

Required the curve in which $\int \frac{d^2y}{dx.dy}$ is a minimum.

[see Euler, *Comm. Acad. Petrop.* tom. VIII. p. 185.]

Here $\frac{d^3y}{dx.dy} = \frac{d^3y}{dx^3} \times \frac{1}{p}.dx$; $\therefore V = \frac{r}{p}$

and $dV = -\frac{r}{p^2}.dp + \frac{1}{p}\,dr$;

$\therefore M=0,\ N=0,\ P=-\frac{r}{p^2},\ Q=0,\ R=\frac{1}{p}$

and by the Form $[f]$ p. 112,

$$\frac{r}{p} = cp + \frac{r}{p} + q.\frac{dp}{dx}.\frac{1}{p^2} \pm c'.$$

Hence, $0 = cp + \frac{q^2}{p^2} - c'$, and $q = p\sqrt{(c'-cp)}$

or, $\frac{dp}{dx} = p\sqrt{(c'-cp)}$ and $\therefore dx = \frac{dp}{p\sqrt{(c'-cp)}}$

and $x = \frac{1}{\sqrt{c'}}.\text{h. l.}\left(\frac{\sqrt{c'}-\sqrt{(c'-cp)}}{\sqrt{c'}+\sqrt{(c'-cp)}}\right)$.

Prob. 11.

Required the relation of x and y when $\int\frac{(d^2y)^2}{dx^3}$ is a maximum, [see Lacroix, *Diff. Calc.* p. 704: also Borda *Mem. Acad. des Sciences*, 1767. p. 560.]

$\frac{(d^2y)^2}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^2.dx = q^2.dx$; $\therefore V = q^2$,

and $dV = 2q.dq$; $\therefore M=0,\ N=0,\ P=0,\ Q=2q,\ R=0$, &c.

$\therefore$ by the Form $[a]$ p. 110, $\frac{d^2Q}{dx^2}=0$, or $2.\frac{d^2q}{dx^2}=0$;

and integrating $2.\frac{dq}{dx} = c$, $2q$ or $2\frac{dp}{dx} = cx + c'$

$2p$ or $2\frac{dy}{dx} = \frac{cx^2}{2} + c'x + c''$, and

$$2y = \frac{cx^3}{2.3} + \frac{c'x^2}{1.2} + c''x + c'''.$$

The final equation contains four arbitrary quantities, c, c', &c.: in order to determine them, suppose the values of y to be given at two points of the abscissa ($x=0$, $x=a$), and also the angles in which the tangents to the curve at those points * are inclined to the axis; for the sake of simplicity, suppose $y=0$, when $x=0$; and $y=b$, when $x=a$: also the values of the two tangents of the angles in which the curve cuts the axis to be t, t': then we have, for the determination of c, c', &c. the following equations:

$$x = 0,\ y = 0; \quad \therefore\ c''' = 0,$$

$$x = a,\ y = b; \quad \therefore\ 2b = \frac{ca^3}{2.3} + \frac{c'.a^2}{2} + c''a,$$

$$x = 0;\ \therefore \frac{dy}{dx} = t; \quad \therefore\ 2t = c'',$$

$$x = a;\ \therefore \frac{dy}{dx} = t'; \quad \therefore\ 2t' = \frac{ca^2}{2} + c'a + c'',$$

whence the values of c, c'.

The preceding problems involve one property only; that of the maximum or minimum; and therefore, in strictness, ought not to be classed amongst Isoperimetrical problems, since they involve neither the Isoperimetrical property, properly so called, nor any other equally affecting the theory and the analytical processes. The following problems involve more than one property.

* This is only one of the many hypotheses that may be framed for the determination of the arbitrary quantities.

Prob. 12.

Required the brachystochrone, that is, the curve of quickest descent, when the length of the curve is given. [See John Bernoulli's Works; tom. II. p. 255. *Acad. des Sciences*, 1718. p. 120; also of this Work, pp. 18, &c.]

Here, by the formulæ, pp. 78, 109, we must find the variation of $\int dx\,(V - au)$, putting $V - au$ instead of V: now $V = \frac{\sqrt{(1+p^2)}}{\sqrt{y}}$ and $u = \sqrt{(1+p^2)}$;

$$\therefore V - au, \text{ or } V' = \frac{\sqrt{(1+p^2)}}{\sqrt{y}} - a\sqrt{(1+p^2)};$$

$$\therefore dV' = -\frac{1}{2y^{\frac{3}{2}}}\sqrt{(1+p^2)}.dy + \left(\frac{1}{\sqrt{y}} - a\right)\frac{p}{\sqrt{(1+p^2)}}.dp;$$

$$\therefore N = -\frac{1}{2y^{\frac{3}{2}}}\sqrt{(1+p^2)},\ P = \left(\frac{1}{\sqrt{y}} - a\right) \times \frac{p}{\sqrt{(1+p^2)}}$$

$\therefore$ by the Form [*b*] p. 110.

$$\sqrt{(1+p^2)}\left(\frac{1}{\sqrt{y}} - a\right) = \frac{p^2}{\sqrt{(1+p^2)}}\left(\frac{1}{\sqrt{y}} - a\right) + c;$$

or $\left(\frac{1}{\sqrt{y}} - a\right) = c\sqrt{(1+p^2)}$, and by reduction,

$$dx = \frac{c\sqrt{y}.dy}{\sqrt{[1 - 2a\sqrt{y} + (a^2 - c^2)y]}}.$$

If, instead of the length, the area had been given, then $\int u dx = \int y dx$; and consequently, we should have

$$V', \text{ or } V - au = \frac{\sqrt{(1+p^2)}}{\sqrt{y}} - ay,$$

in which case, $P = \frac{p}{\sqrt{y}\sqrt{(1+p^2)}}$,

and, $\frac{\sqrt{(1+p^2)}}{\sqrt{y}} - ay = \frac{p^2}{\sqrt{y}\sqrt{(1+p^2)}} + c$,

whence, $\frac{1}{\sqrt{y}\sqrt{(1+p^2)}} = c + ay$, and by reduction,

$$dx = \frac{(c+ay)\sqrt{y}.dy}{\sqrt{[1-y(c+ay)^2]}}.$$

If $c=0$, $dx = \frac{ay^{\frac{3}{2}}dy}{\sqrt{(1-a^2y^3)}}$ which agrees with Simpson's result, *Fluxions*, p, 492. But, since c is not necessarily $=0$, it follows, that Simpson's solution is not so general as it ought to be.

Prob. 13.

Required the solid of least resistance amongst all the solids of equal capacity. [See Maclaurin's *Fluxions*, p. 751. Emerson's *Fluxions*, p. 188.]

Here, [see Prob. 5. p. 155.], $V = \frac{yp^3}{1+p^2}$, and $\int u\,dx = \int \pi y^2 dx$;

$\therefore V - au$, or $V' = \frac{yp^3}{1+p^2} - ay^2$ (including π in the quantity a.)

Hence, $P = \frac{3yp^2+yp^4}{(1+p^2)^2}$, and by the Form [$b$] p. 110,

$\frac{yp^3}{1+p^2} - ay^2 = \frac{3yp^3+yp^5}{(1+p^2)^2} - c$, whence, by reduction,

$(c-ay^2)\,(1+p^2)^2 = 2yp^3$, or $(c-ay^2)\,ds^4 = 2y\,dy^3.dx$.

If instead of the condition of equal capacity, that of equal superficies be substituted, we have

$V-au$, or $V'=\frac{yp^3}{1+p^2}-ay\sqrt{(1+p^2)}$,

and $P=\frac{3yp^2+yp^4}{(1+p^2)^2}-\frac{ayp}{\sqrt{(1+p^2)}}$,

whence, by the Form [*b*] p. 110,

$$\frac{yp^3}{1+p^2}-ay\sqrt{(1+p^2)}=\frac{3yp^3+yp^5}{(1+p^2)^2}-\frac{ayp^2}{\sqrt{(1+p^2)}}-c,$$

and, by reduction,

$$c.(1+p^2)^{\frac{3}{2}}=2yp^3+ay.(1+p^2)^{\frac{5}{2}}$$

$$\text{or } c.ds^4=2y.dy^3.dx+ay.ds^3.dx.$$

If the solid of least resistance be required with both the specified conditions, then we must use a formula $\int(V-au-bv)\,dx$. See p. 78, and if $u=y^2$, and $v=y\sqrt{(1+p^2)}$, we have

$$V-au-bv, \text{ or } V'=\frac{yp^3}{1+p^2}-ay^2-by\sqrt{(1+p^2)},$$

and consequently, by the Form [*b*] p. 110.

$$V'=\frac{3yp^3+yp^5}{(1+p^2)^2}-\frac{byp^2}{\sqrt{(1+p^2)}}-c, \text{ whence}$$

$$c(1+p^2)^2=2yp^3+by(1+p^2)^{\frac{5}{2}}+ay^2(1+p^2)^2,$$

$$\text{or } (c-ay^2)ds^4=2ydy^3dx+bydy^3.dx.$$

Prob. 14.

Given the length of the curve, to determine its nature when the solid generated by its rotation is a maximum. [See Simpson, p. 486: Maclaurin's *Fluxions*, p. 749; Euler, *Methodus Inveniendi*, &c. p. 196.]

$\int V dx=\int \pi y^2.dx$ [$\pi=3.14159$ &c.] and $\int u dx=\int y^2 dx$;

$$\therefore V', \text{ or } V-au=\pi y^2-a\sqrt{(1+p^2)},$$

and $dV' = 2\pi y.dy - \frac{ap.dp}{\sqrt{(1+p^2)}}$, whence $P = -\frac{ap}{\sqrt{(1+p^2)}}$;

$\therefore$ by the Form [*b*] p. 110.

$$\pi y^2 - a\sqrt{(1+p^2)} = c - \frac{ap^2}{\sqrt{(1+p^2)}},$$

and $\therefore \sqrt{(1+p^2)} = \frac{a}{\pi y^2 - c}$, and by reduction,

$$p, \text{ or } \frac{dy}{dx} = \frac{\sqrt{[a^2-(\pi y^2-c)^2]}}{\pi y^2 - c}, \text{ or } dx = \frac{(\pi y^2 - c)}{\sqrt{[a^2-(\pi y^2-c)^2]}}dy.$$

If $c = 0$, $dx = \frac{\pi y^2}{\sqrt{(a^2-\pi^2 y^4)}}dy$, which is Simpson's result, and, like the preceding, restricted.

Euler says, that the curve is the elastic; and the curve determined by Simpson is the rectangular elastic curve.

Prob. 15.

Required the curve that generates the solid of the least surface, the area being given. [See Euler, *Methodus Inveniendi*, &c. p. 198.]

$$V = 2\pi y\sqrt{(1+p^2)}, \quad u = y;$$

$$\therefore V', \text{ or } V - au = 2\pi y\sqrt{(1+p^2)} - ay,$$

and $dV' = [2\pi\sqrt{(1+p^2)} - a]\,dy + \frac{2\pi yp}{\sqrt{(1+p^2)}}.dp,$

whence $P = \frac{2\pi yp}{\sqrt{(1+p^2)}}$; $\therefore$ by the Form [*b*] p. 110,

$$2\pi y\sqrt{(1+p^2)} - ay = \frac{2\pi yp^2}{\sqrt{(1+p^2)}} + c,$$

and, by reduction,

$$p, \text{ or } \frac{dy}{dx} = \frac{\sqrt{[4\pi^2y^2 - (c+ay)^2]}}{c+ay},$$

$$\text{or } dx = \frac{c+ay}{\sqrt{[4\pi^2y^2-(c+ay)^2]}}.dy.$$

If instead of the area, the length is given, then $\int u dx = \int ds = \int dx\sqrt{(1+p^2)}$; and $u = \sqrt{(1+p^2)}$; $\therefore V' = 2\pi y\sqrt{(1+p^2)} - a\sqrt{(1+p^2)}$, and consequently

$$P = \frac{2\pi yp - ap}{\sqrt{(1+p^2)}} \text{ and by the Form } [b] \text{ p. 110,}$$

$$(2\pi y - a)\sqrt{(1+p^2)} = \frac{(2\pi y - a)p^2}{\sqrt{(1+p^2)}} + c,$$

whence, by reduction,

$$dx = \frac{c\,dy}{\sqrt{[(2\pi y-a)^2-c^2]}},$$

and $x = \frac{c}{2\pi}.\text{hyp. log.}[(2\pi y-a)+\sqrt{(2\pi y-a)^2-c^2}]+c'$,

and the curve is the catenary. [See Euler, *Methodus Inveniendi*, p. 198.]

Prob. 16.

Given the length of a curve; required its nature, when its area is a maximum. [See p. 41.]

Here $V' = \sqrt{(1+p^2)} - ay$; $\therefore P = \frac{p}{\sqrt{(1+p^2)}}$, and by the Form $[b]$ p. 110,

$$\sqrt{(1+p^2)} - ay = \frac{p^2}{\sqrt{(1+p^2)}} + c; \therefore \frac{1}{\sqrt{(1+p^2)}} = c+ay,$$

and p, or $\frac{dy}{dx} = \frac{\sqrt{[1-(c+ay)^2]}}{(c+ay)}$, hence,

$$dx = \frac{c+ay}{\sqrt{[1-(c+ay)^2]}}.dy, \text{ and } x = c' - \sqrt{[1-(c+ay)^2]}$$

an equation to a circle.

Prob. 17.

If from a point two straight lines be drawn, inflected to each other, and their extremities be joined by a curve; it is required to find its nature when its length is a maximum; the area included within the two lines and the curve being given. [See Euler, *Methodus Inveniendi*, &c. p. 136.]

Let a line drawn from the fixed point to the curve, (usually called a radius vector) $= y$, and let $d\theta$, (rad. $= 1$,) be the elementary angle between two contiguous lines, y and $y+dy$, then the arc $= \int \sqrt{(dy^2 + y^2.d\theta^2)}$. But the differential of the area $= \frac{y^2.d\theta}{2} = \frac{dx}{2}$, dx being invariable; $\therefore\ d\theta^2 = \frac{dx^2}{y^4}$; and substituting

$\int \sqrt{\left(dy^2 + \frac{dx^2}{y^2}\right)}$ = a maximum, or putting $p = \frac{dy}{dx}$,

$\int \sqrt{\left(p^2 + \frac{1}{y^2}\right)}.dx$ = a maximum. Hence, $V = \sqrt{\left(p^2 + \frac{1}{y^2}\right)}$, and $P = \frac{yp}{\sqrt{(y^2p^2+1)}}$,

and by the Form [*b*] p. 110,

$\sqrt{\left(p^2 + \frac{1}{y^2}\right)} = \frac{yp^2}{\sqrt{(y^2p^2+1)}} + c$, or $1 = cy\sqrt{(y^2p^2+1)}$;

Hence p, or $\frac{dy}{dx} = \frac{\sqrt{(1-c^2y^2)}}{cy^2}$; $d\theta = \frac{dx}{y^2} = \frac{c.dy}{\sqrt{(1-c^2y^2)}}$

the property of a circular arc described on a diameter =

$\frac{1}{c}$; the fixed point, from which lines such as y are drawn, being in one extremity of the diameter.

Prob. 18.

Required the curve that, by a revolution round its axis, generates the greatest solid under a given surface. [See Euler, *Methodus Inveniendi*, p. 194.]

Here V', or $V - au = \pi y^2 - 2a\pi y\sqrt{(1+p^2)}$;

$\therefore P = -\frac{2a\pi yp}{\sqrt{(1+p^2)}}$, and by the Form [$b$] p. 110,

$$\pi y^2 - 2a\pi y\sqrt{(1+p^2)} = -\frac{2a\pi yp^2}{\sqrt{(1+p^2)}} + c\pi,$$

putting $c\pi$ for the correction: hence, by reduction,

$$p \text{ or } \frac{dy}{dx} = \frac{\sqrt{[4a^2y^2-(y^2-c)^2]}}{y^2-c};$$

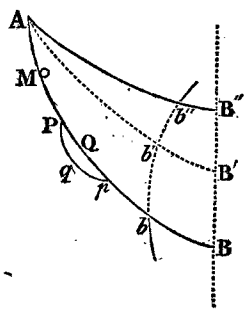
$$\therefore dx = \frac{(y^2-c).dy}{\sqrt{[4a^2y^2-(y^2-c)^2]}} \text{ the equation to the curve.}$$

If $c=0$, $dx = \frac{ydy}{\sqrt{(4a^2-y^2)}}$, an equation to a circle: which Simpson [*Fluxions*, p. 487], determines it to be; but, it is plain that this is only a particular case of the general solution. Lyons's *Solution*, p. 100, is also equally restricted with Simpson's.

Prob. 19.

Of all Isochronous curves to find AB such, that the space included between the arc AB, and a chord drawn from A to B shall be a maximum. [See John Bernoulli's Works, tom. II. p. 263; also *Acad. Roy. des Sciences*, 1718, p. 132.]

Here the time, a given quantity represented by $\int u dx$, answers to $\int dx \frac{\sqrt{(1+p^2)}}{\sqrt{y}}$,



and $\int V dx$, the maximum, to

$$\int \left[y dx - d\left(\frac{xy}{2}\right)\right], \text{ or } \int \left(\frac{y dx}{2} - \frac{x dy}{2}\right),$$

or $\int \left(\frac{y}{2} - \frac{xp}{2}\right) dx$, since $p = \frac{dy}{dx}$.

Hence, $V - au$, or $V' = \frac{1}{2}(y - xp) - a\frac{\sqrt{(1+p^2)}}{\sqrt{y}}$;

$$\therefore M = -\frac{p}{2},\ N = \frac{1}{2} + \frac{a}{2}\frac{\sqrt{(1+p^2)}}{y^{\frac{1}{2}}}, \text{ and } P = -\frac{x}{2} - \frac{ap}{\sqrt{y}\sqrt{(1+p^2)}};$$

$\therefore$ by the Form [c] p. 110,

$$\frac{1}{2}(y - xp) - a\frac{\sqrt{(1+p^2)}}{\sqrt{y}} = -\frac{xp}{2} - \frac{ap^2}{\sqrt{y}\cdot\sqrt{(1+p^2)}} - \int \frac{p dx}{2} + c.$$

But $p = \frac{dy}{dx}$; $\therefore \int p dx = y$. Hence, by reduction,

$y - \frac{a}{\sqrt{y}\sqrt{(1+p^2)}} = c$, and p, or $\frac{dy}{dx} = \frac{\sqrt{[a^2 - y(y-c)^2]}}{\sqrt{y}.(y-c)}$,

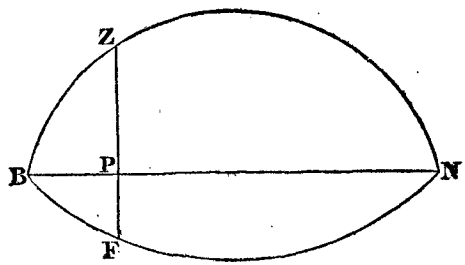
and consequently,

$$dx = \frac{\sqrt{y}.(y-c)dy}{\sqrt{[a^2 - y.(y-c)^2]}},$$

the equation to the curve.

PROB. 20.

Of all Isoperimetrical curves drawn between B and N, to find BFN such, that BZN shall contain the great-



est area, PZ being a function of PF. [See James and John Bernoulli's Works, p. 909, and p. 816; also pp. 12, 25, of this Work.]

Let $PF = y$, and let $PZ =$ a function of $y = Y$; then $\int V dx$ corresponds to $\int Y dx$, and $\int u dx$ to $\int \sqrt{(1+p^2)}.dx$.

Hence, $V + au$, or $V' = Y + a\sqrt{(1+p^2)}$

and $dV' = \frac{dY}{dy}dy + \frac{ap}{\sqrt{(1+p^2)}}.dp$; $\therefore P = \frac{ap}{\sqrt{(1+p^2)}}$

Hence by the Form [b] p. 110,

$$Y + a\sqrt{(1+p^2)} = \frac{ap^2}{\sqrt{(1+p^2)}} + c,$$

or, $\frac{a}{\sqrt{(1+p^2)}} = c - Y$, and by reduction,

p, or $\frac{dy}{dx} = \frac{\sqrt{[a^2-(c-Y)^2]}}{c-Y}$, and consequently,

$$dx = \frac{(c-Y)dy}{\sqrt{[a^2-(c-Y)^2]}}$$ the equation to the curve.

In the equation, as it stands, there are two arbitrary undetermined quantities a and c. The integration of the equation would introduce a third: and to determine these three, we have given, the two points B and N, and the length of the curve.

Prob. 21.

Required the curve which generates by its rotation the solid of the greatest volume; the length of the curve and its area being given. [See Lacroix, *Calc. Diff. et Int.* vol. II. p. 713.]

Here we must use the form [See p. 78.]

$\int(V - au - bv).dx$, corresponding to which is

$$\int[y^2 - ay - b.\sqrt{(1+p^2)}]\,dx,$$

a and b involving, in their values, π and other given quantities.

Hence, the differential of the quantity corresponding to V is

$$(2y - a).dy - bp.\frac{dp}{\sqrt{(1+p^2)}};$$

$\therefore$ by the Form [b] p. 110,

$$y^2 - ay - b\sqrt{(1+p^2)} = c - \frac{bp^2}{\sqrt{(1+p^2)}};$$

$$\therefore y^2 - ay - c = \frac{b}{\sqrt{(1+p^2)}},$$ and by reduction,

$$p, \text{ or } \frac{dy}{dx} = \frac{\sqrt{[b^2-(y^2-ay-c)^2]}}{y^2-ay-c}, \text{ and}$$

$$dx = \frac{(y^2-ay-c)\,dy}{\sqrt{[b^2-(y^2-ay-c)^2]}},$$

an equation to the elastic curve.

PROB. 22.

Let now PZ, instead of being a function of y, (see Fig. p. 130.) be a function of the arc BF (s). [See James and John Bernoulli's Works, p. 912, and p. 917: *Maclaurin's Fluxions*, p. 508, also p. 25. of this Work.]

Here $\int V dx$ corresponds to $\int Z dx$, Z a function of s, and $\int u dx$ corresponds to $\int dx\sqrt{(1+p^2)}$;

$$\therefore V = au, \text{ or } V' = -a\sqrt{(1+p^2)} + Z,$$

$$\text{and } dV' = -\frac{ap}{\sqrt{(1+p^2)}}.dp + \frac{dZ}{ds}.ds.$$

This form therefore is to be solved by the method given in p. 93: and, on comparison, we have $\mu = 0$, $\nu = 0$, $\pi = -\frac{ap}{\sqrt{(1+p^2)}}$, $T = \frac{dZ}{ds}$, $ds = dt$. The formula of solution then, p. 93, is

$$ad\left(\frac{p}{\sqrt{(1+p^2)}}\right) + d\left(\frac{kp}{\sqrt{(1+p^2)}}\right) = 0,$$

k being $= H - \int T dx$. Hence, integrating

$$\frac{ap+kp}{\sqrt{(1+p^2)}} = c, \text{ or } ap + kp = c\sqrt{(1+p^2)}$$

$$\therefore k = \frac{c\sqrt{(1+p^2)} - ap}{p}, \text{ and } dk = \frac{c\,dp}{p^2\sqrt{(1+p^2)}} = -T dx;$$

$$\therefore T dx\sqrt{(1+p^2)}, \text{ or } T ds = -c.\frac{dp}{p^2}: \therefore \int T ds, \text{ that is,}$$

$\int dZ$ or $Z = c' + \frac{c}{p}$. Hence p or $\frac{dy}{dx} = \frac{c}{Z - c'}$, and

$$dx = dy \frac{(Z - c')}{c}.$$

If dx, dy are to be expressed in terms of the arcs, then since $dx = dy . \frac{(Z - c')}{c}$, $ds = \sqrt{\left[1 + \left(\frac{Z - c'}{c}\right)^2\right]} . dy$

consequently, $dy = \frac{c . ds}{\sqrt{[c^2 + (Z - c')^2]}}$

$$dx = \frac{(Z - c') . ds}{\sqrt{[c^2 + (Z - c')^2]}}.$$

PROB. 23.

Given the length of the curve; required its nature when its center of gravity is most remote from the axis (y). [See Euler, *Comm. Acad. Petrop.* tom. VI. p. 146: Simpson's *Fluxions*, p. 498.

Suppose y to be horizontal, and x vertical, then the distance of the center of gravity from y, $= \frac{\int x ds}{\int ds}$ *.

But $\int ds = s = b$ by hypothesis;

$\therefore V - au$, or $V' = \frac{x . \sqrt{(1 + p^2)}}{b} - a\sqrt{(1 + p^2)}$,

and $P = \left(\frac{x}{b} - a\right) \times \frac{p}{\sqrt{(1 + p^2)}}$.

Hence, by the general formula, $N - \frac{dP}{dx}$ + &c. N, Q,

* Simpson's *Fluxions*, p. 205. ed. 1750.

&c. being = 0, we have

$$d\left[\left(\frac{x}{b} - a\right)\frac{p}{\sqrt{(1+p^2)}}\right] = 0;$$

$$\therefore \frac{x}{b} - a = c\frac{\sqrt{(1+p^2)}}{p}, \text{ and } dx = bc \times \frac{-dp}{p^2\sqrt{(1+p^2)}}.$$

Hence, $dx\sqrt{(1+p^2)}$, or $ds = -cb.\frac{dp}{p^2}$, and integrating,

$$s = \frac{cb}{p} + c', \text{ or } p = \frac{cb}{s-c'}; \therefore \text{ since } \sqrt{(1+p^2)} = \frac{ds}{dx},$$

$$dx = \frac{(s-c')\,ds}{\sqrt{[(s-c')^2 + c^2b^2]}},$$

$$\text{and } dy = \frac{cb.ds}{\sqrt{[(s-c')^2 + c^2b^2]}},$$

equations to the catenary.

If the condition of equal length be omitted, that is, if $a = 0$, P will equal $\frac{x}{b}\frac{p}{\sqrt{(1+p^2)}}$; and the nature of the resulting curve will still remain the same, s being represented by b: and if s be not so represented, but be supposed to vary, still the resulting curve will be the catenary, as will appear, with the reason thereof, in the following problem.

Prob. 24.

Suppose a chain or cord, of variable thickness, to be attached to two fixed points; it is required to find the nature of the curve which it ought to form, so that its center of gravity shall be the lowest.

Let the weight of an element of the curve be expressed by $\theta.ds$, θ being a function of s dependent on the

law of the chain's thickness; then, the distance of the center of gravity, instead of being $\frac{\int x ds}{\int ds}$, is $\frac{\int x\theta . ds}{\int \theta . ds}$, which is to be a maximum.

Hence, $\delta\left(\frac{\int x\theta ds}{\int \theta ds}\right) = \frac{\int\theta . ds . \delta \int x\theta ds - \int x\theta . ds . \delta \int \theta . ds}{(\int \theta . ds)^2} = 0$,

now $\delta \int x\theta . ds = \delta \int x\theta \sqrt{(1+p^2)} . dx$; $\therefore V = x\theta\sqrt{(1+p^2)}$

and $dV =$

$$\theta\sqrt{(1+p^2)}.dx + \frac{\theta x p . dp}{\sqrt{(1+p^2)}} + x\sqrt{(1+p^2)}.\frac{d\theta}{ds}.ds.$$

Hence by the Form [*C*] p. 93, supposing $\delta x = 0$, and not regarding *S*, the sum of the definite terms,

$$\delta \int V dx = \int dx . \delta y \left[- d\left(\frac{\theta x p}{\sqrt{(1+p^2)}}\right)\frac{1}{dx}\right] + \int dx . \delta y \left[- d\left(\frac{kp}{\sqrt{(1+p^2)}}\right)\frac{1}{dx}\right] \quad [m]$$

since $v = 0$, $\pi = \frac{\theta x p}{\sqrt{(1+p^2)}}$, $N = 0$, $P = \frac{p}{\sqrt{(1+p^2)}}$, and k, which equals $H - \int T dx$, being in this case $= H - \int x\sqrt{(1+p^2)}.\frac{d\theta}{ds}.dx = H - \int x . d\theta = H - x\theta + \int \theta . dx$

substituting this value in the expression [*m*]; the equation becomes

$$\delta \int V dx = \int dx . \delta y . \left[- d\left(\frac{Hp + p\int \theta . dx}{\sqrt{(1+p^2)}}\right).\frac{1}{dx}\right].$$

With regard to the other variation $\delta \int \theta . ds$, *V* answers to $\theta . \sqrt{(1+p^2)}$, and $dV = \frac{\theta p dp}{\sqrt{(1+p^2)}} + \sqrt{(1+p^2)}.\frac{d\theta}{ds} ds$;

$\therefore$ by Form [C] p. 93, supposing $\delta x = 0$, and not regarding S the sum of the definite parts, $\delta \int V dx =$

$$\int dx . \delta y \left[-d . \frac{\theta p}{\sqrt{(1+p^2)}} \frac{1}{dx} - d . \frac{kp}{\sqrt{(1+p^2)}} \frac{1}{dx} \right] \ldots [n]$$

v being $= 0$, $\pi = \frac{\theta p}{\sqrt{(1+p^2)}}$, and k in this case being equal to $K - \int \sqrt{(1+p^2)} . \frac{d\theta}{ds} . dx$; that is, $K - \int d\theta$, or $K - \theta$; K is analogous to H, and equals the entire value of θ between the two limiting values of x. Hence, substituting in [n] l. 3, we have

$$\delta \int \theta . ds = \int dx . \delta y \left[- d \left(\frac{Kp}{\sqrt{(1+p^2)}} \right) \frac{1}{dx} \right] .$$

If we now substitute in the expression for $\delta \left(\frac{\int x\theta . ds}{\int \theta ds} \right)$ p. 135. l. 4, and represent the two integrals of $x\theta\, ds$, and $\theta\, ds$, taken between the two limiting values [x_0 and x_1] of x, by A and B, there results

$$B \int dx \delta y \left[- d \left(\frac{Hp + p \int \theta\, dx}{\sqrt{(1+p^2)}} \right) . \frac{1}{dx} \right] -$$

$$A \int dx\, \delta y \left[- d \left(\frac{Kp}{\sqrt{(1+p^2)}} \right) . \frac{1}{dx} \right] = 0,$$

and putting $\frac{A}{B} = \lambda$,

$d \left(\frac{Hp + p \int \theta dx}{\sqrt{(1+p^2)}} \right) = \lambda . d \left(\frac{Kp}{\sqrt{(1+p^2)}} \right)$, and integrating

$Hp + p \int \theta dx = \lambda Kp + c \sqrt{(1+p^2)}$, whence

$H-\lambda K+\int\theta dx=c\frac{\sqrt{(1+p^2)}}{p}$, and taking the differential

$\theta dx=-c\frac{dp}{p^2\sqrt{(1+p^2)}}$, or $\theta dx\sqrt{(1+p^2)}=-c\frac{dp}{p^2}$,

or $\theta ds=-c\frac{dp}{p^2}$; whence integrating

$\int\theta ds=\frac{c}{p}+c'$, and p or $\frac{dy}{dx}=\frac{c}{\int\theta ds-c'}$,

and since $\sqrt{(1+p^2)}=\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}=\frac{ds}{dx}$

we have $dx=\frac{\int\theta ds-c'}{\sqrt{[(\int\theta ds-c')^2+c^2]}}\cdot ds$,

and $dy=\frac{cds}{\sqrt{[(\int\theta ds-c')^2+c^2]}}$.

If the chain be uniformly thick, $\theta=1$ and

$$dx=\frac{(s-c')\,ds}{\sqrt{[(s-c')^2+c^2]}},$$

$$dy=\frac{cds}{\sqrt{[(s-c')^2+c^2]}},$$

which equations are the same, in fact, as those in the preceding problem, [p. 134: l. 67], and belong to the curve called the catenary.

In problem the twenty-third, where s is a given quantity, the solution was effected by means of the common formula $N-\frac{dP}{dx}+$&c.; and the same formula is sufficient, if instead of the process there used, we institute one similar to that in Prob. 24, p. 134, that is, if we

investigate the variation of $\frac{\int x ds}{\int ds} - a \int ds$, which equals

$$\frac{s \times \delta \int x ds - \int x ds \delta \int ds}{s^2} - a \delta \int ds$$

or $\frac{1}{s^2}[s.\delta \int [x \sqrt{(1+p^2)}] dx - (\int x ds + as^2) \delta \int \sqrt{(1+p^2)} dx]$

for, it is manifest, that this expression can be expanded by the common formula, p. 88 *.

The condition however of a variable thickness introduces the quantity θ, a function of s; for instance, instead of $\delta \int x \sqrt{(1+p^2)}.dx$ – &c., we have, [see p. 135. l. 5.] $\delta \int x \theta \sqrt{(1+p^2)}$ – &c.; and therefore when the differential of the quantity corresponding to V is taken, a term $x \sqrt{(1+p^2)} \frac{d\theta}{ds}.ds$ is introduced, which obliges us to have recourse to the formula [*Cc*] p. 93.

The next instance will illustrate the use of Lagrange's formula in his third problem, p. 185, in which, the quantity that is to be a maximum is expressed under the form of a differential equation.

Prob. 25.

Required the curve, down which a body falling, in a resisting medium, shall acquire the greatest velocity.

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* Euler, *Comm. Acad.* tom. VI. shews, that the catenary equally results, whether the length of the curve be taken into the computation or not. "Potuisset quidem eadem æquatio multo facilius inveniri, si in $\frac{\int x ds}{s}$ neglexissem denominatorem, quippe per priorem conditionem debet in omnibus curvis esse idem: Verum quia hoc fortuito accidit, &c.

Let z represent the square of the velocity, g the force of gravity, ϱ a function of z, the resistance; and let x be supposed vertical: then,

$$dz - 2g.dx + 2\varrho\sqrt{(1+p^2)}.dx = 0, \ldots [1]$$

[by the principles of mechanics.]

Compare this with the formula p. 94. l. 12, and we have $V=z$, $X=2g$, $U=2\varrho\sqrt{(1+p^2)}$ a function of ϱ and p;

$$\therefore dU = \frac{dU}{dp}dp + \frac{dU}{d\varrho}.d\varrho = \frac{2\varrho p}{\sqrt{(1+p^2)}}.dp +$$

$$2\frac{d\varrho}{dz}.dz\sqrt{(1+p^2)}$$

$$\therefore d\phi = \frac{2\varrho p}{\sqrt{(1+p^2)}}.dp \text{ and } T = 2\frac{d\varrho}{dz}\sqrt{(1+p^2)}:$$

hence, the equations are,

$$\frac{d\lambda}{dx} - 2\lambda.\frac{d\varrho}{dz}\sqrt{(1+p^2)} = 0 \ldots [2]$$

and δV or $\delta z = 0 = e^{-\int T dx}\int e^{T dx}.\delta\phi = 0$,

$$\text{or } 2\lambda\varrho\frac{p}{\sqrt{(1+p^2)}} = \frac{1}{\sqrt{a}} \ldots [3]$$

In equation [2] substitute from [1] the value of dz;

$$\therefore 2g d\lambda = 2\lambda\sqrt{(1+p^2)}.d\varrho + 2\varrho\sqrt{(1+p^2)}.d\lambda$$

$$= 2\sqrt{(1+p^2)}[\lambda d\varrho + \varrho.d\lambda] = 2\sqrt{(1+p^2)}\,d(\lambda\varrho) =$$

$$2.d[\lambda\varrho\sqrt{(1+p^2)}] - 2\lambda\varrho.\frac{p\,dp}{\sqrt{(1+p^2)}} = \text{(by [3])}$$

$$2.d[\lambda\varrho\sqrt{(1+p^2)}] - \frac{dp}{\sqrt{a}};$$

Hence, integrating, $2g\lambda = 2\lambda\varrho\sqrt{(1+p^2)} - \frac{p}{\sqrt{a}} + c =$

$$\frac{1+p^2}{p\sqrt{a}} - \frac{p}{\sqrt{a}} + c = \frac{1}{\sqrt{a}.p} + c;$$

and $\therefore$ from [3], $\frac{c}{g}\varrho\frac{p}{\sqrt{(1+p^2)}} + \frac{\varrho}{g\sqrt{a}\sqrt{(1+p^2)}} = \frac{1}{\sqrt{a}}$:

whence, by the solution of a quadratic the value of p.

This problem may be also solved, and readily, by means of the general method and formula given in p. 104, thus: since z is to be a maximum, $\int V dx$ corresponds to

$$\int \frac{dz}{dx} dx;$$

$\therefore dV = d\left(\frac{dz}{dx}\right)$; $\therefore$ by comparison, $\pi = 1$, all the other coefficients being $= 0$; $\therefore Y = 0$, $Z = 0$, $Y' = 0$, $Z' = 1$. Again,

$\int W dx$ corresponds to $\int \left[\frac{dz}{dx} - 2g + 2\varrho . \sqrt{(1+p^2)}\right] dx$

$\therefore$ by comparison,

$$P' = \frac{2\varrho p}{\sqrt{(1+p^2)}}, \quad \nu' = 2\sqrt{(1+p^2)}\,\frac{d\varrho}{dz}, \quad \pi' = 1;$$

$$\therefore (Y) = \lambda N' - \frac{d(\lambda P')}{dx} = -d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right)\frac{1}{dx},$$

$$(Y') = \lambda P' = \frac{2\lambda\varrho p}{\sqrt{(1+p^2)}},$$

$$(Z) = \lambda\nu' - \frac{d(\lambda\pi')}{dx} = 2\lambda\sqrt{(1+p^2)}.\frac{d\varrho}{dz} - \frac{d\lambda}{dx},$$

$$(Z') = \lambda\pi' = \lambda.$$

Hence, taking from the expanded form, for $\delta\int V dx + \delta\int \lambda W dx$, or

$$\delta\int \frac{dz}{dx} dx + \delta\int \left[\lambda\frac{dz}{dx} - 2g\lambda + 2\lambda\varrho\sqrt{(1+p^2)}\right] dx, \text{ that}$$

part which is under the sign $\int$, and on which the nature of the curve depends, we have

$(Y).\delta\omega + (Z')\delta\omega' = 0$, and $(Y)=0$, $(Z)=0$,

since $\delta\omega$, $\delta\omega'$ are independent of each other; hence,

$$d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right)=0, \text{ and } 2\lambda\frac{d\varrho}{dz}\sqrt{(1+p^2)}-\frac{d\lambda}{dx}=0$$

as before, p. 139. l. 12, 14.

The following problem will illustrate the use of the general method given in p. 104, in which the maximum or minimum property involves a quantity not expressed except by means of a differential equation; which equation, in the subjoined case, is called an equation of condition.

Prob. 26.

Required the brachystochrone, or the curve of quickest descent, in a medium resisting as any function of the velocity.

Let z represent the square of the velocity, g the force of gravity, ϱ a function of z, the resistance; and let y be horizontal and x vertical: then by the principles of mechanics,

$$dz - 2g.dx + 2\varrho.ds = 0,$$

$$\text{or } \left[\frac{dz}{dx}-2g+2\varrho\sqrt{(1+p^2)}\right]dx = 0$$

the differential equation of condition.

If we compare this with Wdx, p. 104, we have

$$dW = 2.\frac{d\varrho}{dz}\sqrt{(1+p^2)}.dz + d\left(\frac{dz}{dx}\right) + 2\varrho.\frac{pdp}{\sqrt{(1+p^2)}};$$

$\therefore \nu' = 2.\frac{d\varrho}{dz}\sqrt{(1+p^2)}$, $\pi = 1$, since $d\left(\frac{dz}{dx}\right) = dp'$,

and $P' = \frac{2\varrho p}{\sqrt{(1+p^2)}}$;

$\therefore (Y) = -d\frac{(\lambda P')}{dx} = -d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right)\frac{1}{dx}$;

and $(Z) = \lambda\nu' - d\frac{(\lambda\pi')}{dx} = 2\lambda\sqrt{(1+p^2)}\cdot\frac{d\varrho}{dz} - \frac{d\lambda}{dx}$.

The equation of minimum corresponding to $\int V dx$ is $\int\frac{\sqrt{(1+p^2)}}{\sqrt{z}}.dx$;

$\therefore dV = \frac{p\,dp}{\sqrt{z}\sqrt{(1+p^2)}} - \frac{1}{2}\frac{\sqrt{(1+p^2)}}{z^{\frac{3}{2}}}.dz$;

and by comparison,

$N=0$, $P = \frac{p}{\sqrt{z}\sqrt{(1+p^2)}}$, $Q=0$, &c. $\nu = -\frac{\sqrt{(1+p^2)}}{2z^{\frac{3}{2}}}$,

and $Y = N - \frac{dP}{dx} + \&c. = -d\left(\frac{p}{\sqrt{z}\sqrt{(1+p^2)}}\right)\frac{1}{dx}$

$Z = \nu - \frac{d\pi}{dx} + \&c. = -\frac{\sqrt{(1+p^2)}}{2z^{\frac{3}{2}}}$.

Hence, substituting in the expression for $\delta\int V dx + \delta\int(\lambda W)\,dx$ p. 104, and retaining only the parts under the integral sign, which must separately equal 0, we have

$Y + (Y) = 0$, or $-d\left(\frac{p}{\sqrt{z}\sqrt{(1+p^2)}}\right) - d\left(\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\right) = 0$,

$Z + (Z) = 0$, or $-\frac{\sqrt{(1+p^2)}}{2z^{\frac{3}{2}}} + 2\lambda\sqrt{(1+p^2)}.\frac{d\varrho}{dz} - \frac{d\lambda}{dx} = 0$.

If we put $t = \frac{1}{\sqrt{z}} + 2\lambda\varrho$, and integrate the first of these two equations, it becomes

$$\frac{tp}{\sqrt{(1+p^2)}} = \frac{1}{\sqrt{a}} \quad \ldots\ldots\ldots \quad [1]$$

the second is $\frac{d\lambda}{dx} - \left(2\lambda\frac{d\varrho}{dz} - \frac{1}{2z^{\frac{3}{2}}}\right)\sqrt{(1+p^2)} = 0 \;\ldots\; [2]$

Add to these $dz - 2g.dx + 2\varrho.dx\sqrt{(1+p^2)} = 0 \;\ldots\; [3]$ and the equation to the curve is to be determined from these three.

From the value of t, $\frac{dt}{dz} = -\frac{1}{2z^{\frac{3}{2}}} + 2\frac{d\lambda}{dz}.\varrho + 2\lambda\frac{d\varrho}{dz}$;

and from this equation substitute the value of $-\frac{1}{2z^{\frac{3}{2}}} + 2\lambda\frac{d\varrho}{dz}$, into [2], and there results

$$\frac{d\lambda}{dx} + \left(2\varrho\frac{d\lambda}{dz} - \frac{dt}{dz}\right)\sqrt{(1+p^2)} = 0.$$

In this, substitute the value of dz derived from [3], and there results (two quantities destroying each other)

$$2g.\frac{d\lambda}{dx} - \frac{dt}{dx}\sqrt{(1+p^2)} = 0 \;\ldots\; [a]$$

but $\frac{dt}{dx}\sqrt{(1+p^2)}.dx = d\left[t\sqrt{(1+p^2)}\right] - t\frac{p\,dp}{\sqrt{(1+p^2)}}$

$$= d\left[t\sqrt{(1+p^2)}\right] - \frac{dp}{\sqrt{a}},$$

(from the value of t in equation [1];)

hence, the equation [a] is

$$2g.\frac{d\lambda}{dx} - d\frac{\left[t\sqrt{(1+p^2)}\right]}{dx} + \frac{dp}{dx\sqrt{a}}; \text{ and integrating}$$

$$2g\lambda - t\sqrt{(1+p^2)} + \frac{p}{\sqrt{a}} = c;$$

or, substituting the value of t from [1] p. 143,

$$2g\lambda - \frac{(1+p^2)}{p\sqrt{a}} + \frac{p}{\sqrt{a}} = c;$$

$$\text{or, } 2g\lambda = \frac{1}{p\sqrt{a}} + c.$$

Hence, $t = \frac{1}{\sqrt{z}} + 2\lambda\varrho = \frac{1}{\sqrt{z}} + \frac{\varrho}{g}\left(\frac{1}{p\sqrt{a}} + c\right) = \frac{\sqrt{(1+p^2)}}{p\sqrt{a}}$, from equation [1], and consequently,

$$\frac{1}{\sqrt{z}} = \frac{\sqrt{(1+p^2)}}{p\sqrt{a}} - \frac{\varrho}{g}\left(\frac{1}{p\sqrt{a}} + c\right);$$

from this equation obtain the value of dz, which will involve the differentials dp, and $\frac{d\varrho}{dz}.dz$. By means of equation [3], eliminate dz, and there will result a differential equation involving dx and dp, or $\frac{d^2y}{dx^2}.dx$: which will be the equation of solution.

In the case of a void, or non-resisting medium $\varrho = 0$;

$$\therefore t = \frac{1}{\sqrt{z}} = \frac{\sqrt{(1+p^2)}}{p\sqrt{a}}, \text{ or } \frac{p}{\sqrt{(1+p^2)}\sqrt{z}} = \frac{1}{\sqrt{a}},$$

or $\frac{dy}{ds\sqrt{z}} = \frac{1}{\sqrt{a}}$, an equation to the cycloid. [See pp. 5, 21.]

The preceding instances are sufficient, it is hoped, fully to illustrate the nature and use of those formulæ of solution invented by Euler, on which the nature of the curve, or the relation of x to y, depends. But hitherto, no illustration has been given of those other formulæ,

that, on assigning the initial and final variations, have definite and absolute values *. These formulæ, it has been already said, were invented by Lagrange, and their use will be shewn in the following cases, which may be viewed, as unresolved parts, or undetermined conditions belonging to certain of the preceding problems.

Prob. 27.

If the brachystochrone is to lie between a point and a curve, required the angle in which the cycloid must cut the curve.

By Form [*Aa*] p. 89, and since the quantity under the integral sign $\int$, $= 0$,

$$V_1.\delta x_1 + P_1.\delta \omega_1 = 0;$$

for, the initial point is given; $\therefore \delta x_0, \delta y_0 = 0$, and $\delta \omega_0 = \delta y_0 - p_0 \delta x_0, = 0$; and since

$$V = \frac{\sqrt{(1+p^2)}}{\sqrt{y}}, \quad Q = 0, \text{ and } P = \frac{p}{\sqrt{y}.\sqrt{(1+p^2)}},$$

[see p. 113].

Hence, $(V_1 - P_1 p_1).\delta x_1 + P_1.\delta y_1 = 0$,

but $V = Pp + c$ [p. 110]; $\therefore c.\delta x_1 + P_1.\delta y_1 = 0$ . . [*m*].

Let now M (a function of x_1 and y_1) $= 0$; then, [see p. 107.]

$$\frac{dM}{dx}.\delta x_1 + \frac{dM}{dy}.\delta y_1 = 0, \text{ and } \frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0.$$

Eliminate δx_1, δy_1 by means of this and of the preceding equation [*m*], and there results

* P. 89, 90, 91.

$$\frac{dy_1}{dx_1} = -\frac{c}{P_1} = -\frac{V-Pp}{P_1} = -\frac{1}{p_1}.$$

Now $p = \frac{dy}{dx}$, y and x belonging to the cycloid, and p_1 is what $\frac{dy}{dx}$ becomes at the limiting curve, when for x and y, x_1 and y_1 are substituted: but, $\frac{dy}{dx}$ is the tangent of the angle which a tangent line to the curve makes with the axis *, and $\frac{dx}{dy}$ or $\frac{1}{p}$ is the co-tangent of the same angle †, and $\frac{1}{p_1}$ is the co-tangent at the limiting curve: for the same reason $\frac{dy_1}{dx_1}$, is the tangent of the angle which a tangent line to the limiting curve makes with the axis: and since $\frac{dy_1}{dx_1} = -\frac{1}{p_1}$, or since, the tangent of the latter angle equals the co-tangent of the former, one angle is the complement of the other: their sum therefore is equal to a right angle‡ consequently, the remaining angle of the triangle, or that which the two tangents form, is a right angle; or, the two curves intersect each other at right angles. [See p. 11.]

The same conclusion will follow from the expression for the tangent of the difference of two angles [*A*, *B*] the

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* *Princ. Anal. Calc.* p. 173. † *Trig.* p. 9.

denominator of which expression is $1 + \tan. A . \tan. B$ *, and in the symbols previously employed, $1 + p_1 \cdot \frac{dy_1}{dx_1} = 0$;

$\therefore$ the fraction or the tangent $(A - B)$, is ∞; or the difference of the two angles is a right angle.

PROB. 28.

Suppose now the brachystochrone to lie, not between a point and a curve, but between two curves. Required the angles in which it must cut them.

By Form [*Aa*] p. 89.

$$V_1 . \delta x_1 - V_0 . \delta x_0 + P_1 . \delta \omega_1 - P_0 . \delta \omega_0 = 0;$$

$\therefore$ as before $c \delta x_1 + P_1 . \delta y_1 - (c . \delta x_0 + P_0 . \delta y_0) = 0$, and this equation, if there be no relation between the two limiting curves, or between x_0, y_0, and x_1, y_1, resolves itself into these two,

$c . \delta x_1 + P_1 . \delta y_1 = 0$ [1], and $c . \delta x_0 + P_0 . \delta y_0 = 0$ .. [2]

If therefore L, (a function of x_0, y_0) $= 0$, be the equation to the initial limiting curve, by combining it with [2] we shall arrive at the same results as were obtained in p. 145, by combining $M = 0$ with [1]; since the respective equations are similar. Hence, the brachystochrone, which here is a cycloid, must cut the first curve also at right angles. [See *Misc. Taurin.* tom. IV. p. 187.]

In this last case the origin of the abscissas is supposed to be fixed, and consequently, the velocity with which the

* Woodhouse's *Trigonometry*, p. 23.

body leaves the first curve is variable, varying as $\sqrt{y_0}$. If the origin of the abscissas is supposed to vary, then, [see p. 108.] terms such as $-\delta a\int\frac{dV}{da}, -\delta b\int\frac{dV}{db}$, will be introduced into the equation of limits: thus, instead of the differential of the time being $\frac{ds}{\sqrt{y}}$, let it be $\frac{ds}{\sqrt{(y-h)}}$, and let h be a function of a $[x_0]$, and b $[y_0]$ solely; then,

since, $V=\frac{\sqrt{(1+p^2)}}{\sqrt{(y-h)}}$, $\frac{dV}{da}=\frac{\sqrt{(1+p^2)}}{2.(y-h)^{\frac{3}{2}}}.\frac{dh}{da}$, and

$$\frac{dV}{db}=\frac{\sqrt{(1+p^2)}}{2(y-h)^{\frac{3}{2}}}.\frac{dh}{db}.$$

In this case therefore, the equation of limits, at the last point where $x=x_1$, or $=a'$, since $\frac{dh}{da'}=0$, and $\frac{dh}{db'}=0$ [h not involving a' and b'] becomes,

$$c\delta x_1 + P_1.\delta y_1 = 0 \ldots\ldots [3]$$

but, the equation of limits, at the first point, becomes

$$c.\delta x_0+P_0.\delta y_0+\left(\frac{dh}{da}\delta x_0+\frac{dh}{db}.\delta y_0\right)\int\frac{ds}{2.(y-h)^{\frac{3}{2}}}=0, [4]$$

Since by the gen^{l}. form^{a}. $N-\frac{dP}{dx}=0$; $\therefore \frac{ds}{2(y-h)^{\frac{3}{2}}}=-dP$,

and $\int\frac{ds}{2.(y-h)^{\frac{3}{2}}}=-P+$ corr. Now at the beginning, the integral $=0$, and $P=P_0$; $\therefore$ corr. $=P_0$. Hence,

$\int\frac{ds}{2(y-h)^{\frac{3}{2}}}=P_0-P=P_0-P_1$ [from $x=x_0$ to $x=x_1$]

the equation [4] therefore becomes

$$\left[c - \frac{dh}{da}(P_0 - P_1)\right]\delta x_0 + \left[P_0 - \frac{dh}{db}\left(P_0 - P\right)\right]\delta y_0 = 0.$$

This last result, expressed in the equations [3] and [4], comprehends, as particular cases, the preceding ones p. 146, 147; for if $h=0$, the equations [3] and [4] become,

$$c.\delta x_1 + P_1.\delta y_1 = 0, \text{ and } c.\delta x_0 + P_0\delta y_0 = 0$$

as before; which equations shew that the cycloid must cut the two curves at right angles. This was Lagrange's original determination of the conditions. [See *Misc. Taurin.* tom. II. p. 180.]

If $h=b$, or $b-e$, [e a constant quantity;) that is, if $y-h$, at the point of departure, $=0$ or $=e$; or if, in other words, the velocity of the body when it quits the first curve is either nothing or constant; or, if the origin of the abscissas, according to M. Borda's expression, be supposed to be situated in the first curve: then, $\frac{dh}{db}=1$, and $\frac{dh}{da}=0$; $\therefore$ the equation [4] becomes

$c.\delta x_0 + P_1.\delta y_0 = 0$. But the equation [3] is

$c.\delta x_1 + P_1.\delta y_1 = 0$;

consequently, if from the equations $L=0$, $M=0$, we eliminate, as before p. 145, δx_0, δx_1, &c. there will result from the first $\frac{dy_0}{dx_0} = -\frac{c}{P_1}$, and from the second $\frac{dy_1}{dx_1} = -\frac{c}{P_1}$; consequently, $\frac{dy_0}{dx_0} = \frac{dy_1}{dx_1}$, or the tangent at

the point of intersection of the cycloid and first curve, is parallel to the tangent, at the intersection of the cycloid and second curve. This is the result which M. Borda first arrived at [*Mem. Acad. des Sciences*, 1767, p. 558,] and which was afterwards confirmed by Lagrange, and Legendre. [*Misc. Taurin.* tom. IV. p. 187; and *Mem. Acad. des Sciences*, 1786, p. 30.]

This completes the solution of the brachystochrone, of which there are six cases, [see pp. 113, 122, 141, 145, 147, 149.] and this curve which first called the attention of mathematicians to the connected doctrine and calculus, has been also the object of their latest researches.

Newton, in the *Acta Eruditorum*, Mai 1697, p. 223* gave, without proof or the authority of his name, a method of describing the cycloid. But John Bernoulli, from the Work recognised its author: "*ex ungue Leonem*" †.

As additional instances of the use of the equation of limits, we will solve the following problems.

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* See also *Phil. Trans.* No. 224. p. 384: his Works by Horsley, vol. IV. p. 415: and his Opuscula, vol. I. p. 289.

† Quoique l'auteur de cette construction par un exces de modestie ne se nomme pas, nous savons pourtant indubitablement par plusieurs circumstances que c'est le celebre Newton, et quand meme nous ne le saurions point d'ailleurs, ce seroit assez de le connoitre par un echantillon, comme *ex ungue Leonem*, Joann. Bernoulli, *Opera*, tom. I. p. 197.

Prob. 29.

Required the conditions for drawing a right line that is to be, the shortest distance, between two given curves.

1. By Prob. 2. p. 113, the curve was determined to be a straight line; also $V = \sqrt{(1 + p^2)}$, and $P = \frac{p}{\sqrt{(1 + p^2)}}$.

Hence, by the form, p. 89, the equation of limits becomes

$$V_1.\delta x_1 - V_0.\delta x_0 + P_1.\delta \omega_1 - P_0.\delta \omega_0 = 0,$$

or, since $\delta \omega = \delta y - p.\delta x$ [p. 88.], and

$V - Pp = \frac{1}{\sqrt{(1+p^2)}}$, we have

$$\frac{1}{\sqrt{(1+p_1^2)}}\delta x_1 + P_1.\delta y_1 - \frac{1}{\sqrt{(1+p_0^2)}}\delta x_0 + P_0.\delta y_0 = 0.$$

If x_1, x_0, &c. are independent of each other, this equation resolves itself into

$$\frac{\delta x_1}{\sqrt{(1+p_1^2)}} + P_1.\delta y_1 = 0, \text{ and } \frac{\delta x_0}{\sqrt{(1+p_0^2)}} + P_0.\delta y_0 = 0.$$

If $M = 0$, and $L = 0$, are the two equations to the two limiting curves, there results, as before, p. 145.

$$\frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0, \quad \frac{dy_0}{dx_0}\delta x_0 - \delta y_0 = 0.$$

Hence, by means of these, and of the two preceding equations, eliminating δx_1, δx_0, &c. there results

$$\frac{dy_1}{dx_1} = -\frac{1}{p_1}, \text{ and } \frac{dy_0}{dx_0} = -\frac{1}{p_0};$$

and consequently, for the reasons assigned in p. 146, the line must intersect each curve at right angles.

Prob. 30.

It is required to investigate the conditions for the curve, down which, a body, in a resisting medium, acquires the greatest velocity. [See p. 138.]

We have, p. 104, the equation of limits,

$$0 = (V + \lambda W).\delta x + [Y' + (Y')] \times (\delta y - p\delta x) + [Z' + (Z)]\,(\delta z - p'\delta x),$$

in which [see p. 140.]

$$V = \frac{dz}{dx},\quad W = \frac{dz}{dx} - 2g + 2\varrho\sqrt{(1+p^2)},$$

$$Y' = 0,\quad (Y') = \frac{2\lambda\varrho p}{\sqrt{(1+p^2)}},\quad Z' = 1,\quad (Z) = \lambda.$$

Hence, the equation of limits is

$$0 = \left[\frac{dz}{dx}(1+\lambda) - 2g\lambda + 2\varrho\lambda\sqrt{(1+p^2)}\right]\delta x +$$

$$\frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}\,(\delta y - p\delta x) + (1+\lambda)\left(\delta z - \frac{dz}{dx}\right)\delta x,$$

and, by reduction,

$$\left(-2g\lambda + \frac{2\lambda\varrho}{\sqrt{(1+p^2)}}\right)\delta x + \frac{2\lambda\varrho p}{\sqrt{(1+p^2)}}.\delta y + (1+\lambda).\delta z = 0,$$

and if from p. 139, we substitute the values of $2g\lambda$ and $2\lambda\varrho$ there results

$$-c\delta x + \frac{\delta y}{\sqrt{a}} + (1+\lambda).\delta z = 0,$$

which is the general expression: and consequently, the two equations at the limits, x_0, x_1, become

$$-c\delta x_0 + \frac{1}{\sqrt{a}}.\delta y_0 + (1+\lambda_0).\delta z_0 = 0,$$

$$-c.\delta x_1 + \frac{1}{\sqrt{a}}.\delta y_1 + (1 + \lambda_1).\delta z_1 = 0.$$

Now z is the square of the velocity; and if the initial velocity be given, $\delta z_0 = 0$; in the second equation, since δz_1 is indeterminate, make $(1 + \lambda_1)\,\delta z_1 = 0$, or $\lambda_1 = -1$, and then the two equations are reduced to

$$-c.\delta x_0 + \frac{1}{\sqrt{a}}.\delta y_0 = 0,$$

$$-c.\delta x_1 + \frac{1}{\sqrt{a}}.\delta y_1 = 0.$$

If $L=0$, and $M=0$, are the two equations belonging to the limiting curves; then there results, as before, p. 145.

$$\frac{dy_0}{dx_0}.\delta x_0 - \delta y_0 = 0,$$

$$\frac{dy_1}{dx_1}.\delta x_1 - \delta y_1 = 0.$$

From these two last, and the two preceding, eliminating δx_0, δy_0, &c. there results,

$$\frac{dy_0}{dx_0} = c\sqrt{a}, \quad \frac{dy_1}{dx_1} = c\sqrt{a},$$

consequently, the tangents at the respective points of intersection of the curve, and the two limiting curves, are parallel to each other.

The conditions relative to the curve of quickest descent may be similarly determined. The preceding instances are, however, sufficient to explain the use of Lagrange's determinate formulæ.

The purpose of the present treatise, is now accomplished. There are, however, several curious results deducible from Euler's formula; but, to these, since they are not closely connected with the matter of the preceding pages, the reader is merely referred*.

---

* Lagrange, *Fonctions Analy.* p. 217, &c. *Leçons sur le Calcul des Fonctions,* p. 401, &c. Lacroix, *Calcul. Diff.* &c. p. 660, 667, &c. Lexell. *Novi Comm. Petrop.* tom. XV. p. 127, &c.

THE END.

A

TREATISE

ON

PLANE AND SPHERICAL

Trigonometry.

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THE SECOND EDITION,

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PREFACE

TO THE SECOND EDITION.

It is not easy to adapt a Treatise on Trigonometry to all descriptions of Students; to state, in its beginning, within a small compass, and with their simplest solutions, those Propositions which relate merely to the cases of oblique-angled triangles, and, then, on the ground of those propositions and by the method of their solutions, to proceed to investigations of greater intricacy.

The Student, if he be supposed to possess a knowledge of the first six Books of Euclid, may thence, by a few easy inferences, and by the aid of some simple constructions, arrive most readily at the Trigonometrical solutions of the cases of oblique-angled triangles. If his views extend no farther, he cannot take a better guide than Ludlam or Robert Simson; nor proceed by any easier method than the *Geometrical*.

But, few Students are content with such confined views. Trigonometry is now extended far beyond its original object, and to other investigations than those of the relations of the sides and angles of Triangles. The *collateral* uses of the science have become the most numerous, and are not the least important. To the knowledge of many of these, however, the Geometrical

method is unable to conduct us. At some point or other of our enquiries (we speak of its present and actual state) it must be abandoned, and recourse be had to that which technically is called the *Analytical Method.*

Since this latter is the sole thoroughly efficient method, will it not be better to make it, in a Treatise on Trigonometry, the predominant one, instead of being compelled merely to call in its aid, when the resources of the former are exhausted?

The Author of the present Treatise has endeavoured to construct it on such a plan; and, with this view, he has had as little recourse as possible, to Geometrical constructions and the properties of figures. What he thence has borrowed are not so much to be considered as the first steps in his process of demonstration, as the data and ground-work from which the process itself is to commence and to be instituted.

By these means the process is made uniform and systematical. But, uniformity may be purchased at too dear a rate; and the main purpose of the Work, which is utility, would be sacrificed, if, for the sake of system, the analytical method were reluctantly compelled to submit to modes of proof that are strange to its nature and genius.

The specimens of demonstration contained in the following pages must determine whether or not such sacrifice has been made.

The great practical use of Trigonometry is the resolution of rectilinear triangles; but, that it is capable of being extended, and, to objects, not merely curious, but of real interest, we may learn from the history and actual state of the science.

The first considerable extension of Trigonometry, beyond its original object, was made about twenty years after the death of

Newton. Then, on the ground-work laid down by that great man, three Mathematicians of the Continent, Clairaut, Dalembert and Euler, and Thomas Simpson our countryman, began to establish a system of Physical Astronomy more perfect than what its Author had left. With this view, they laid aside the Geometrical method which Newton had used, and which they thought incompetent to their purpose, and adopted the Analytical. Pursuing this method, they perceived the formulæ of Trigonometry to be of continual use and recurrence, and, the language by which the process of demonstration was conducted, to be formed, in a great degree, of symbols and phrases borrowed from that science. In order, therefore, to render the process precise and expeditious, it became necessary to improve the means and instruments by which it was carried on; and, accordingly, at the time spoken of, the advancement of Trigonometry, the pure and subsidiary science, was contemporaneous with that of Astronomy, the mixed and principal one.

This general statement would be confirmed by an examination of the Memoirs and Treatises on Physical Astronomy published about the year 1750.

Clairaut and Dalembert in their *Lunar Theories* embody in those Works, or introduce as prefatory matter, several, now commonly known, Trigonometrical formulæ*. In the Volume of Tracts which Thomas Simpson published, the Author evidently

* It will hardly be believed that Theorems, such as are given in pp. 25, 26, &c. were almost unknown. Yet Clairaut, (*Mem. Acad.* 1745, p. 342, and *Theorie de la Lune*, Edit. 2, p. 9,) alluding to these Theorems, says, 'M. Euler est le premier, que Je sçache, qui ait faite usage de ces Theoremes pour operer sur les sinus et cosinus d'angles, sans avoir recours a leurs formes imaginaires.'

intended the one which is inserted at p. 76, as preparatory to the succeeding Theory of the Moon; and, Euler distinctly states as a reason for cultivating the algorithm of sines, its great utility in the mixed Mathematics.

In the arrangement of the Treatise, which the Table of Contents sufficiently explains, Spherical succeeds to Plane Trigonometry. Now, the former has not, like the latter, been extended beyond its original purpose. It has no collateral and indirect uses; it has not enriched the general language of analysis, by its peculiar phrases. But, notwithstanding this confined range, and apparent simplicity in the object of the science, its propositions are more easily established by the Analytical method than the Geometrical. And, (at least in the opinion of the Author of this Treatise) this would be the case, even if there existed no similarity and artificial connexion, between the processes by which the series of formulæ in the two branches of Trigonometry were respectively established. But, so far from there being no similarity, the corresponding propositions can be deduced by methods so analogous, that to know the one is almost to know the other.

This will appear to be the case, if we refer to pages 21 and 128, &c. of this Treatise. We shall there find similar Algebraical derivations of formulæ from two fundamental expressions for the cosine of an angle. The principle of the derivation, however, is not new; it originated with Euler, who inserted in the *Acta Acad. Petrop.* for 1779, a Memoir entitled *Trigonometria Spherica Universa, ex primis principiis breviter et delucide derivata.* Gua next, in the *Memoirs of the Academy of Sciences* for 1783, p. 291, deduced, but by awkward and complicated processes, Spherical Trigonometry "from the Algebraical solution of the simplest of its Problems." In 1786, Cagnoli, in his excellent Treatise, derived without "similar triangles or complicated figures," the

fundamental expressions for the sine and cosine of the sum of two arcs. And lastly, Lagrange and Legendre, the one in the *Journal de L'Ecole Polytechnique*, the other in his *Elemens de Geometrie*, have followed and simplified Euler's method, and instead of three fundamental expressions, have shewn one to be sufficient.

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$$\sin.(A + B) + \sin.(A - B) = 2 \sin. A. \cos. B$$
$$\sin.(A + B) - \sin.(A - B) = 2 \cos. A. \sin. B$$
$$\sin.(A + B). \sin.(A - B) = \sin.^2 A - \sin.^2 B$$
$$\cos.(A - B) + \cos.(A + B) = 2 \cos. A. \cos. B$$
$$\cos.(A - B) - \cos.(A + B) = 2 \sin. A. \sin. B$$
$$\sin. A \pm \sin. B = 2 \sin. \frac{A \pm B}{2} . \cos. \frac{A \mp B}{2}$$
$$\cos. A + \cos. B = 2 \cos. \frac{A + B}{2} . \cos. \frac{A - B}{2}$$

23—28

$$\cos. B - \cos. A = 2 \sin. \frac{A+B}{2} . \sin. \frac{A-B}{2}$$

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}} \ldots\ldots$$

$$\frac{\cos. A + \cos. B}{\cos. B - \cos. A} = \frac{1}{\tan. \frac{A+B}{2} . \tan. \frac{A-B}{2}} \ldots\ldots$$

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \frac{A+B}{2} \ldots\ldots$$

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PLANE TRIGONOMETRY.

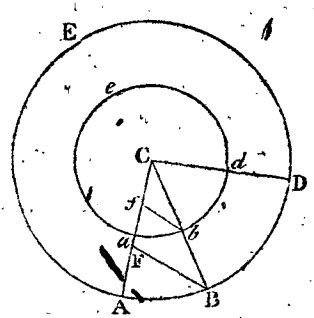
CHAP. I.

On the Division of the Circle into Degrees, Minutes, Seconds, &c.— Definitions of Sines, Cosines, &c.

ARCS THE MEASURES OF ANGLES.

ART. 1. IT is proved in the 33d Proposition of the sixth Book of Euclid, that, in equal circles, angles have the same ratio to each other, as the arcs on which they stand. Hence also, in the same circle, the angles vary as the arcs on which they stand; and consequently we may assume arcs as the measures of angles.

In the circle *ABDE*, the arcs *AB*, *AD*, are measures of the angles *ACB*, *ACD*; and of the same angles, in the smaller circle *abde*, *ab*, *ad*, are the measures; which latter arcs have the same



ratio to the arcs *AB*, *AD*, that the radius *Ca* has to *CA*. For, since, in the circle *abde*, the measure of four right angles is the whole circumference *abde*,

$\angle aCb : 4$ right $\angle s :: ab : abde$, therefore

$$\angle aCb = 4 \text{ right } \angle s \times \frac{ab}{abde}$$

Similarly, $\angle ACB$, or $\angle aCb = 4$ right $\angle s \times \frac{AB}{ABDE}$

Hence, $\frac{ab}{AB} = \frac{abde}{ABDE} = \frac{Ca}{CA}$, since, (Playfair's *Geometry*, Edit. 2. p. 219.) the circumferences of circles are to one another as their radii.

2. If from the points B, b, two lines BF, bf be drawn making with CA equal angles CFB, Cfb, then, by the similar triangles CFB, Cfb, we have $\frac{BF}{bf} = \frac{CB}{Cb}$

and hence, if, in a circle $ABDE$, we have determined the value of a line such as BF, we can always assign the value of a similar line bf, in another circle $abde$, provided the ratio of the radii CB, Cb, be known: for instance, if CB be called 1, and Cb, r, $bf = BF \times \frac{r}{1} = BF \times r$, or $BF = \frac{bf}{r}$.

3. * It is usual to divide the circumference of a circle into 360 equal parts, which parts are called degrees, and of which the symbol is $n°$ or $5°$, if n or 5 be their number: each degree is also divided into 60 equal parts, which parts are called minutes; and of which the symbol is m' or $7'$, if m or 7 be their number: and, finally, each minute is divided into 60 equal parts, which parts are called seconds, and of which the symbol is t'' or $35''$, if t or 35 be their number: thus, if AB (Fig. p. 1.) equals one-fourth of the circle $ABDE$, AB contains 90 degrees, or symbolically, $AB = 90°$. If $AB = \frac{1}{12}$th of the circumference $ABDE$, AB contains 30° or $AB = 30°$. If $AB = \frac{1}{7}$th of the circumference $ABDE$, $AB = \frac{1}{7}360° = 51° + \frac{3}{7}1° = 51° + \frac{3.60'}{7} = 51° 25' + \frac{5}{7}1' =$ $51° 25' 42'' + \frac{6}{7}1''$.

* See the Note at the end of the Chapter.

The values of $\frac{6}{7}$ 1″, and of like quantities are, usually, expressed by means of decimals; thus, $\frac{6}{7}$ 1″, retaining only the two first figures, equals 0″.85; and $\frac{1}{7}$th of the circumference would be expressed by

$$51^\circ\ 25'\ 42''.85.$$

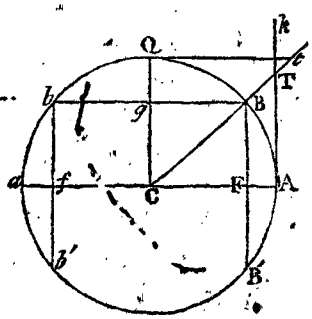
But, it is occasionally useful (see *Astron.* p. 397.) to extend the division of the circle beyond that of seconds, and to introduce, with their proper symbols, *thirds*, *fourths*, &c. In such an extension, $\frac{6}{7}$ 1″ would equal $51'''\ 25^{IV}.7$ and the foregoing arc, the seventh of the circumference, would be expressed by

$$51^\circ\ 25'\ 42''\ 51'''\ 25^{IV}.7.$$

4. The arcs of circles, it has appeared, are proper measures of the angles which they subtend; if the angles be increased, the arcs are also increased, and in the same ratio; and knowing the value of one, is, in fact, knowing the value of the other. But, in Trigonometry, the values of angles are made to depend on the values of certain right lines, drawn according to certain rules, but not varying as the angles vary. The lines just alluded to are called sines, tangents, secants, &c. which it now becomes necessary to define.

Sine.

If from the extremity (A) of any arc such as AB, a diameter Aa be drawn, and on it a perpendicular BF be let fall, from the



other extremity B, such perpendicular is called the *Sine* of the

arc AB. And, if AB be represented by the character A, the sine BF is, usually, thus represented, sin. A.

Agreeably to the above definition, bf (perpendicular to Aa) is the sine of Ab, and $b'f$ the sine of Abb'.

Since BFB', bfb', are perpendicular to the diameter Aa, $BF=B'F$, and $bf=b'f$ (*Euclid*, Book III. Prop. 3.); also, the arc $AB=$arc AB', and the arc $ab=$arc ab'. For, conceive two right lines to be drawn joining A, B, and A, B'; then, since in the triangles AFB, AFB', $BF=B'F$, AF is common, and $\angle AFB=\angle AFB'$, the right line AB, the side of the $\triangle AFB$, must equal the right line AB', the side of the $\triangle AFB'$, and, accordingly, (*Euclid*, Book III. Prop. 28.) the arc AB must equal the arc AB'. By the same proof the arc ab must equal the arc ab'. Hence, if the arc AB should equal the arc ab, the arc $B'AB$ would equal the arc $b'ab$; and consequently, the right line BFB' would (*Euclid*, Book III. Prop. 29.) equal the right line bfb'; and, the halves of these lines would be equal, and, BF the sine of the arc AB, would equal bf the sine of the arc Ab, which arc Ab, in this case, equals the semi-circumference $ABba$ minus the arc AB.

In order to express symbolically the preceding equality, let the character π * represent the semi-circle $ABba$, then $Ab=\pi-AB=\pi-A$; consequently, $BF=bf$ may be thus expressed sin. $A=$ sin. $(\pi-A)$; or, if we represent $ABba$ by 180°,

$$\text{sin. } A = \text{sin. } (180° - A)$$

5. $(\pi - A) + A = \pi$, and therefore if we call $\pi - A$ the *supplement* of A, the above equation, sin. $(\pi - A)=$ sin. A, expressed in words, amounts to this proposition: *The sine of the supplement of an arc is equal to the sine of the arc itself.*

If in the form

$$\text{sin. } (\pi - A) = \text{sin. } A,$$

* The numerical value of π is 3,14159, &c.: it is the value of the circumference of a circle, of which the diameter is 1: if, therefore, CA be 1, or the diameter Aa be 2, since circumferences of circles vary as their diameters, the circumference $AbaA$ will be equal to 2π.

in which A may be any arc, we substitute,

instead of A, $\left(\frac{\pi}{2} - A\right)$, we shall have

$$\text{sin.}\left(\frac{\pi}{2} + A\right) = \text{sin.}\left(\frac{\pi}{2} - A\right)$$

and if, instead of A in the same form, we substitute $\frac{m}{n}\pi - A$,

there will result $\text{sin.}\left(\frac{n-m}{n}\pi + A\right) = \text{sin.}\left(\frac{m}{n}\pi - A\right)$.

6. The definition of a sine may be extended so as to designate the sines of arcs that are greater than the circumference. Thus, BF which (see Fig. p. 3.) is the sine of AB, may be considered as the sine of $ABaB + AB$, and as the sine of $2(ABaB) + AB$, &c. and, consequently, the subjoined equations would be true

$\text{sin.}\ A = \text{sin.}\ (2\pi + A) = \text{sin.}\ (4\pi + A)$, and generally,

$$\text{sin.}\ A = \text{sin.}\ (2n\pi + A)$$

n being a whole number.

For like reasons,

$$\text{sin.}\ (\pi - A) = \text{sin.}\ (3\pi - A) = \text{sin.}\ (5\pi - A)$$

and generally,

$$\text{sin.}\ (\pi - A) = \text{sin.}\ \{(2n + 1)\pi - A\}$$

n being any number in the progression, 0, 1, 2, 3, &c. Hence, since by Art. 4. p. 4, $\text{sin.}\ A = \text{sin.}\ (\pi - A)$ there results this general equation

$$\text{sin.}\ (2n\pi + A) = \text{sin.}\ \{(2n + 1)\pi - A\}$$

or, what amounts to the same thing, if s be the sine of any arc A, it is also the sine of all arcs comprehended under the two formulæ

$$(2n\pi + A),\quad \{(2n + 1)\pi - A\},$$

in which n may be any term of the progression, 0, 1, 2, 3, &c.

Versed Sine.

7. The line AF, (Fig. 2.) intercepted between the extremity A of the arc, and the point F the extremity of the sine BF, is called the *versed sine* of the arc AB, and is usually expressed thus, ver. sin. A.

Cosine.

8. The part of the diameter intercepted between the center C and the extremity F of the sine, and which is equal to the difference of the radius CA and of the versed sine AF, is called the *cosine* of the arc AB, and is thus expressed, cos. A : and if r, v, and c, be the characters for the radius, versed sine and cosine, we have this equation, $c = r - v$.

9. If the arc Ab be greater than a quadrant, the versed sine Af is greater than the radius, and the cosine $Cf = v - r$. In the former case therefore, we have $c = r - v$; in the latter, $c = v - r$, and $\therefore -c = r - v$. Hence, if we generally represent the cosine by this equation, $c = r - v$, when v becomes greater than r, that is, when the arc is greater than a quadrant and less than three quadrants, c, in fact, will be negative:

10. If Bg be drawn perpendicularly to CQ, AQ being a quadrant, or $\frac{1}{4}$th of the circle, Bg, by definition, is the sine of BQ. Now, $BQ + AB =$ quadrant, and if, on this account, we denominate BQ the *complement* of AB, we may say that Bg is the sine of the complement of the arc AB. But, $Bg = CF$ and (Art. 8.) CF is the cosine of AB. Hence, the equation $Bg = CF$, translated into other terms, amounts to this proposition: *The Sine of the Complement of an Arc is equal to the Cosine of the same Arc.*

If we express, in symbols, the above equality, we shall have

$$\text{cos. } A = \text{sin. } \left(\frac{\pi}{2} - A\right) = \text{(by Art. 5.) sin. } \left(\frac{\pi}{2} + A\right)$$

in like manner,

$$\text{sin. } A = \text{cos. } \left(\frac{\pi}{2} - A\right) = \text{also, } -\text{cos. } \left(\frac{\pi}{2} + A\right).$$

11. The definition of a cosine may be extended, like that of a sine, to designate the cosines of arcs, that are greater than the circumference. Thus, CF, which is the cosine of AB, may be considered as the cosine of the (circumference + AB), &c. Hence, as before (p. 5.) the following equations will be true; cos. A = cos. $(2\pi + A)$ = cos. $(4\pi + A)$ = cos. $(2n\pi + A)$, n

being any term of the progression, 0, 1, 2, 3, 4, 5, &c. But, since the same CF is also the cosine of the arc $ABaB'$, we have $CF = \cos.(2\pi - A) =$ (for reasons just alledged) $\cos.(4\pi - A) = \cos.(2n\pi - A)$, n being any term of the progression, 1, 2, 3, &c. or, CF, generally, $=$

$$\cos.\left\{(2n+2)\pi - A\right\},$$

n being any term of the progression, 0, 1, 2, 3, &c.

Hence CF is the cosine of all arcs comprehended within the two formulæ

$$(2n\pi + A),\ \left\{(2n+2)\pi - A\right\}.$$

12. In a similar manner we may investigate the general formulæ of arcs that have the same *negative* cosine,

$Cf = \cos.(\pi - A) = \cos.(3\pi - A) =$, generally, $\cos.\left\{(2n+1)\pi - A,\right\}$,

n being any term of the progression, 0, 1, 2, 3, &c.

and since the same Cf is the cosine also of $ABbab'$ or $\pi + A$

$$Cf = \cos.(\pi + A) = \cos.(3\pi + A) = \cos.\left\{(2n+1)\pi + A\right\}$$

Hence Cf is the cosine of all arcs comprehended within the two formulæ

$$\left\{(2n+1)\pi - A\right\},\ \left\{(2n+1)\pi + A\right\}.$$

The arcs of which fb' is the sine, are

$\pi + A$, $3\pi + A$, and generally $\left\{(2n+1)\pi + A\right\}$.

The arcs of which $FB' = fb'$ is the sine, are

$2\pi - A$, $4\pi - A$, and generally $(2n+2)\pi - A$.

Hence FB' is the sine of all arcs comprehended within the two formulæ,

$$\left\{(2n+1)\pi + A\right\} \text{ and } \left\{(2n+2)\pi - A\right\},$$

n being in each case any term of the progression, 0, 1, 2, 3, &c.

In calculations, where FB, FB', and other quantities are involved, if s be the symbol for FB, $-s$ must be the symbol for FB'. For, conceive a line to be drawn a tangent to the circle, at the point opposite to Q in QC produced, and let the distance of any point in the circumference from this line be called z,

then $FB\ (s) = z - r$, and $FB' = r - z$, or $FB' = -(z - r)$ Hence, if in any equation subsisting between trigonometrical lines we wish to pass from the consideration of the point B to that of the point B', we must in such equation substitute $-(z-r)$ instead of $z-r$, or $-s$, instead of s.*

13. The preceding results may be conveniently represented in a Table, s and c representing the sine and cosine of an arc A.

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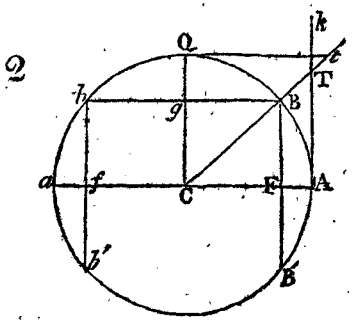
* This hinges on the general doctrine of negative quantities: the scrupulous Student, who is not satisfied with what is here said, is referred to Carnot's *Geometrie de position*, and his subsequent work on the Theory of Transversals, &c.

Values of	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.
n	sine$=s$	sine$=s$	sine$=-s$	sine$=-s$	cos.$=c$	cos.$=c$	cos.$=-c$	cos.$=-c$
0	A	$\pi-A$	$\pi+A$	$2\pi-A$	A	$2\pi-A$	$\pi-A$	$\pi+A$
1	$2\pi+A$	$3\pi-A$	$3\pi+A$	$4\pi-A$	$2\pi+A$	$4\pi-A$	$3\pi-A$	$3\pi+A$
2	$4\pi+A$	$5\pi-A$	$5\pi+A$	$6\pi-A$	$4\pi+A$	$6\pi-A$	$5\pi-A$	$5\pi+A$
3	$6\pi+A$	$7\pi-A$	$7\pi+A$	$8\pi-A$	$6\pi+A$	$8\pi-A$	$7\pi-A$	$7\pi+A$
4	$8\pi+A$	$9\pi-A$	$9\pi+A$	$10\pi-A$	$8\pi+A$	$10\pi-A$	$9\pi-A$	$9\pi+A$
&c.								
General Formulæ	$2n\pi+A$	$(2n+1)\pi-A$	$(2n+1)\pi+A$	$(2n+2)\pi-A$	$2n\pi+A$	$(2n+2)\pi-A$	$(2n+1)\pi-A$	$(2n+1)\pi+A$

Tangent.

14. If from A, ATk be drawn perpendicular to CA; then AT (the point T being the intersection of ATk and of CB produced



through the extremity of the arc AB), is called the *tangent* of AB, and is usually thus expressed, tan. A.

Secant.

CT, intercepted between the center C and the extremity T of the tangent AT, is called the *secant* of AB, and is thus expressed, sec. A.

Co-tangent and Co-secant.

Qt, drawn perpendicularly to CQ, is, by the definition, *l* 2, &c. the tangent of the arc QB; and, since QB (see p. 6.) is the complement of AB, Qt, the tangent of the complement of AB, is called the *co-tangent* of AB (A), and is thus expressed, cot. A. In like manner, Ct which is the secant of QB, and, therefore, the secant of the complement of AB (A) is called the *co-secant* of AB (A) and is thus expressed, co-sec. A.

By the similar triangles CAT, CFB,

$$AT = CA \times \frac{BF}{CF}; \quad \therefore \text{ if } CA = 1, \text{ tan. } A = \frac{\text{sin. } A}{\text{cos. } A};$$

since A may be any arc, substitute in this expression $\frac{\pi}{2} - A$

instead of A; then, since (p. 6.) sin. $\left(\frac{\pi}{2}-A\right)$ = cos. A, and cos. $\left(\frac{\pi}{2}-A\right)$ = sin. A, and (p. 10. l. 12, &c.)

tan. $\left(\frac{\pi}{2}-A\right)$ = co-tan. A, we have

$$\text{co-tan. } A = \frac{\text{cos. } A}{\text{sin. } A}.$$

Hence, tan. A + co-tan. $A = \frac{\text{sin. } A}{\text{cos. } A} + \frac{\text{cos. } A}{\text{sin. } A} = \frac{\text{sin.}^2 A + \text{cos.}^2 A}{\text{sin. } A \text{ cos. } A} = \frac{1}{\text{sin. } A. \text{ cos. } A}$; again, tan. $A \times$ co-tan. $A = \frac{\text{sin. } A}{\text{cos. } A} \times \frac{\text{cos. } A}{\text{sin. } A} = 1.$

15. It would be easy, from the definition of a tangent, to draw it and to shew its variation of magnitude, and change of sign; but, since we have already formulæ and a table for the sine and cosine, it is most convenient to use them and determine the tangent from the expression,

$$\text{tan. } A = \frac{\text{sin. } A}{\text{cos. } A}.$$

For example, when A is very small or 0, the sin. A is, and tan. A is so also.

When A is a quadrant, cos. A is 0, and tan. $A = \frac{\text{sin. } A}{0} = \infty$, or, is said to be infinite.

When A is half a quadrant, or 45°, the sine BF=the cosine CF, and tan. 45° $= \frac{\text{sin. } 45°}{\text{cos. } 45°} = 1$, or the tangent of 45° is equal to the radius.

16. The value of the secant may be had from the similar triangles CAT, CFB, in which $CT = \frac{CB.CA}{CF} = \frac{(CA)^2}{CF}$; or, from the value of the tangent: thus, if r be the radius sec.$^2 A$ =

$$r^2 + \tan.^2 A = r^2 + r^2 \cdot \frac{\sin.^2 A}{\cos.^2 A} = \frac{r^2\{\cos.^2 A + \sin.^2 A\}}{\cos.^2 A}$$

$$= \frac{r^2 \cdot r^2}{\cos.^2 A}$$

consequently, $\sec. A = \frac{r^2}{\cos. A}$, or, if radius $= 1$, $= \frac{1}{\cos. A}$.

The co-secant of A is (see p. 10.) the secant of the complement of A; in symbols, co-sec. A = sec. $\left(\frac{\pi}{2} - A\right)$ = (by the preceding expression) $\frac{1}{\cos. \left(\frac{\pi}{2} - A\right)}$. But, cos. $\left(\frac{\pi}{2} - A\right)$ = sin. A; consequently, the radius being $= 1$,

$$\text{co-sec. } A = \frac{1}{\sin. A}.$$

Hence $\frac{\sec A}{\text{co-sec. } A} = \frac{1}{\cos. A} \times \frac{\sin. A}{1}$ = (by p. 10. l. 18.) tan. A.

Chord.

17. The *chord* of an arc is a straight line joining the extremities of the arc. In the Figure of page 10, BFB' is the chord of BAB', Bgb of BQb, and bfb' of bab'. Hence, since (p. 4. l. 18.) $BFB' = 2BF$ and BF is the sine of BA, which arc $BA = \frac{1}{2}BAB'$, it follows, that the chord of an arc is equal to twice the sine of half that arc.

By *Euclid*, Book III. Prop. 35, $AF \times Fa = BF^2$; and, if we make $AF = v$, and $CA = r$, there will result

$$v \times (2r - v) = \sin.^2 A.$$

Conceive the extremities A, B, of the arc to be joined by the chord AB, then (*Euclid*, Book I. Prop. 47.)

$$\sin.^2 A = (\text{chord } A)^2 - v^2,$$

$$\text{(by l. 14, \&c.)} = 4 \sin.^2 \frac{A}{2} - v^2.$$

Hence,

$$2rv - v^2 = \sin.^2 A = 4 \cdot \sin.^2 \frac{A}{2} - v^2,$$

$$\text{and } \therefore rv = 2 . \sin.^2 \frac{A}{2},$$

$$\text{or (see p. 6.) } r . (r - c) = 2 . \sin.^2 \frac{A}{2}.$$

Transpose this latter equation, and we have it, that is,

$$\cos. A = \frac{1}{r} \left(r^2 - 2 . \sin.^2 \frac{A}{2} \right)$$

$$\text{or, } = \frac{1}{r} \left(2 \cos.^2 \frac{A}{2} - r^2 \right)$$

$$\text{since, } \sin.^2 \frac{A}{2} + \cos.^2 \frac{A}{2} = r^2.$$

Multiply the two latter expressions for cos. A, and

$$\cos.^2 A = \frac{1}{r^2} \left\{ 2r^2 \left(\cos.^2 \frac{A}{2} + \sin.^2 \frac{A}{2} \right) - r^4 - 4 . \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2} \right\}$$

$$= \frac{1}{r^2} \left(r^4 - 4 . \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2} \right)$$

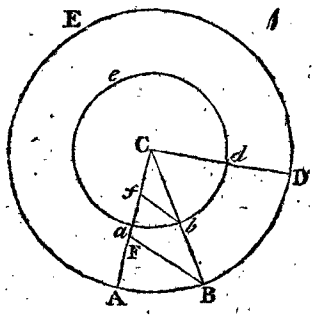
$$\therefore r^2 (r^2 - \cos.^2 A), \text{ or } r^2 . \sin.^2 A = 4 \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2},$$

$$\text{and } \therefore r . \sin. A = 2 \sin. \frac{A}{2} \cos. \frac{A}{2}.$$

18. In some of the preceding expressions, a radius = 1 has been used, and, solely, for the purpose of lessening the number of symbols in the Trigonometrical formulæ. For, 1, or any power or root of it, used as a multiplier or divisor of an expression, may be expunged from such expression; thus, instead of $\frac{\sin.^2 A}{1^2}$, we may more simply write $\sin.^2 A$. Still, however, it is, on many occasions, necessary to use, for the radius, a general symbol such as r, or, an arithmetical value such as 10,000. For this reason, it is desirable to be possessed of some easy and expeditious rule, for converting formulæ constructed with a radius = 1 into other formulæ that shall have a different radius. Such a rule may be obtained from the following simple considerations:

If BF, bf be drawn similarly inclined to CA, then by similar triangles, $BF = bf . \frac{CB}{Cb}$ or $BF = bf . \frac{1}{r}$ (if $CB = 1$, $Cb = r$).

In like manner, $CF = Cf.\frac{1}{r}$, $AF = af\frac{1}{r}$. Now, sines, cosines, tangents, &c. are drawn after the manner that these lines are; if the angle BFC be a right angle, BF is the sine of the angle BCF to radius 1, bf is the sine to radius r; and, CF, Cf,



are the cosines. Hence, in any formula involving sin. A, cos. A, &c. calculated for a radius = 1, substitute instead of sin. A, cos. A; $\frac{\text{sin. } A}{r}$, $\frac{\text{cos. } A}{r}$, and the resulting formula will belong to lines drawn in a circle, of which the radius is r: for instance,

$$\text{tan. } A = \frac{\text{sin. } A}{\text{cos. } A}, \text{ radius} = 1; \therefore \frac{\text{tan. } A}{r} = \frac{\text{sin. } A}{r} \times \frac{r}{\text{cos. } A},$$

$$\text{or, tan } A = r.\frac{\text{sin. } A}{\text{cos. } A}, \text{ (radius} = r).$$

$$\text{Again, sec. } A = \frac{1}{\text{cos. } A} \text{ (radius} = 1), \text{ then}$$

$$\frac{\text{sec. } A}{r} = \frac{1}{\frac{\text{cos. } A}{r}} = \frac{r}{\text{cos. } A}, \text{ or}$$

$$\text{sec. } A = \frac{r^2}{\text{cos. } A}, \text{ (radius} = r).$$

$$\text{Again, tan. } A. \text{ co-tan. } A = 1, \text{ then, } \frac{\text{tan. } A}{r}.\frac{\text{co-tan. } A}{r} = 1,$$

$$\text{and tan. } A. \text{ co-tan. } A = r^2.$$

And if in any formula, any power of sin. A, or of cos. A, such as $\text{sin.}^3 A$, $\text{sin.}^n A$, $\text{cos.}^4 A$, or $\text{cos.}^n A$, occurs, the

radius being 1, by substituting $\frac{\sin.^3 A}{r^3}$, $\frac{\sin.^n A}{r^n}$, $\frac{\cos.^4 A}{r^4}$, $\frac{\cos.^n A}{r^n}$, the resulting formula will belong to a circle of which the radius is r.

This is the rule; but, since it would leave a Trigonometrical formula with fractional terms (the denominators being powers of the radius) it may, with advantage, be modified and made more convenient. Thus, suppose a form should occur such as

$$\cos. nA = a . \cos.^n A + b . \cos.^{n-2} A + \&c. \text{ to radius } 1;$$

then, by what has preceded, the reduced form to a radius r, is

$$\frac{\cos. nA}{r} = \frac{a . \cos.^n A}{r^n} + b . \frac{\cos.^{n-2} A}{r^{n-2}} + \&c.$$

and, cleared of fractions, is

$$r^{n-1} \cos. nA = a . \cos.^n A + b . r^2 . \cos.^{n-2} A + \&c.$$

Here, $\cos. nA$ is multiplied by r^{n-1}, $\cos.^{n-2} A$ by r^2, &c.; that is, if we choose to call $\cos.^n A$ a quantity of n dimensions, $\cos.^{n-2} A$, a quantity of $n-2$ dimensions, $\cos. A \times \sin. A$, a quantity of 2 dimensions, we may announce the preceding rule under the following simple form:

Multiply each term of a Trigonometrical formula, in which the radius $= 1$, *by such power of* r, *as shall make it of the same dimensions with the term of the highest dimensions; the resulting formula will be true when the radius is* $=$ r.

Thus, if

$$\cos. 3A = 4 . \cos.^3 A - 3 . \cos. A \qquad (\text{rad}^s. = 1,)$$

since $\cos.^3 A$ the term of the highest, is of three dimensions, and $\cos. 3A$, $\cos. A$, are of one dimension, we have

$$r^2 . \cos. 3A = 4 . \cos.^3 A - 3r^2 \cos. A.$$

19. The Trigonometrical symbols, such as $\sin. A$, $\tan. A$, &c. that have been obtained, are merely general ones, and, hitherto, no methods have been given of assigning their values in specific values of the angles. The *general* methods for this purpose will be given in a subsequent part of the Treatise; but, even at this stage, by peculiar artifices, we may, in certain simple cases, assign the arithmetical values of the sines and cosines of angles. For example,

If $AB = BQ$, that is, if AB = half a quadrant, or, expressed in degrees, if AB or $\angle ACB = 45°$, since $\angle BCQ = 45°$, also cos. $45°$ = sin. $45°$, but cos.$^2 A$ + sin.$^2 A = 1$, (1 = radius in this case); therefore, $2(\text{sin. } 45°)^2 = 1$, and sin. $45° = \frac{1}{\sqrt{2}} = .7071068$, or, (see the preceding rule) = 7071.068, to a radius = 10,000.

If $ACB = \frac{1}{3} 90° = 30°$, since $BCB' = 2.ACB = 60°$, and since $\angle CBB' = \angle CB'B = 60°$, the triangle BCB' is equilateral; and consequently BB' (the chord of $60°$) = radius $CA = 1$, and, BF = sin. $30° = \frac{1}{2} BB' = \frac{1}{2} = .5$, and $\therefore$ cos. $30°$ or sin. $60°$ (see Art. 10) $= \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2} = .8660254$, and (see p. 14.) =

8660.254 to a radius = 10,000,
and 8660254 to a radius = 1000000.

Hence may be proved, what was asserted in p. 3, that the sines of arcs do not vary as the arcs themselves. For, the sin. $30° = \frac{1}{2}$ radius $= \therefore \frac{1}{2}$ sin. $90°$; in other words, the sines are as 1 to 2, whilst the arcs are as 1 to 3.

The values of the tangent may be found, in the above cases, from the expression tan. $A = \frac{\text{sin. } A}{\text{cos. } A}$. Thus

rad^s. = 1.	rad^s. = 10,000
tan. $45° = \frac{\text{sin. } 45°}{\text{cos. } 45°} = 1$	= 10,000
tan. $30° = \frac{1}{2} \times \frac{2}{\sqrt{3}} = .5773503$	= 5773.503
tan. $60° = \frac{\sqrt{3}}{2} \times \frac{2}{1} = 1.7320508$	= 17320.508
tan. $90° = \frac{\text{sin. } 90°}{\text{cos. } 90°} = \frac{1}{0} = \infty$	

We may here direct the Student's attention to the superior augmentation of the tangent above that of the sine; for, we have, corresponding to

* Arcs	0,	30°	45°	60°	90°
Sines	0,	5000,	7071,	8660,	10000
Tangents	0,	5773,	10000,	17320,	∞.

In the same manner as we have found the values of the tangents, in the preceding cases, we may find those of the versed sine, secant, &c.

In the next Chapter we will proceed to investigate expressions,

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* We have adhered, in this Chapter, to the antient and common division of the circle. But, in most of the French scientific treatises that have, of late years, been published, the circumference is divided first, into 400 equal parts or degrees, then, each degree into 100 equal parts, or minutes, then, each minute into 100 equal parts or seconds: so that a French degree is less than an English in the proportion of 90 to 100: a French minute less than an English, in the proportion of 90×60 to 100×100: and a French second less in the proportion of $90\times60\times60$ to $100\times100\times100$: hence, if n be the number of French degrees, the corresponding number of English equals $\frac{n\,.\,9}{10}$ or $\frac{n(10-1)}{10}$ or $n-\frac{n}{10}$, which last form points to an easy arithmetical operation for finding the number of degrees in the English scale from the number in the French scale: since from the proposed number we must subtract the same, after the decimal point has been moved one place to the left:

Examples: What number of degrees, minutes, &c. in the English scale correspond to 73°, to 71° 15′, and to 26°.0735, in the French scale.

73	71 . 15	26.0735
7.3	7 . 115	2.60735
65.7	64 . 035	23.46615
6	6	6
65° 42′ English	2.10	27 . 9690
	6	6
	6	58.140 Answer 23° 27′ 58″
	64° 2′ 6″	

for the sine and cosine of the sum and difference of two arcs, in terms of the sine and cosine of the simple arcs. Such expressions are, in this science, very important, since from them, by an easy derivation, may be made to flow almost all other Trigonometrical formulæ.

This operation of reducing French to English degrees may be superseded, and rendered less liable to mistake, by means of the opposite Table, in which, as it is usual, the reduction is effected simply by addition.

Example to the Table.

Reduce 26°.0735 to English degrees, &c.

	French.				English.			
By Table,	20°	0′	0″	.....	18°	0′	0″	
	6	0	0	.....	5	24	0	
	0	.07	0	.....	0	3	46.8	
	0	0	30	.....	0	0	9.72	
	0	0	5	.....	0	0	1.62	
	26	.0735			23	27	58.14	the same as before.

Reduce 2°.7483 to English degrees, &c.

	French.				English.		
By Table,	2°	0′	0″	.....	1°	48′	0″
	0	.70	0	.....	0	37	48
	0	4	0	.....	0	2	9.6
	0	0	80	.....	0	0	25.92
	0	0	3	.....	0	0	.972
	2°	.7483		.....	2	28	24.492

TABLE

For Reducing French Degrees, &c. to English.

French Division:

400° in the circle, 100′ in a degree, 100″ in a minute.

English Division:

360°.....60′.....60″

	French.	English.	French.	English.
Degrees.	1°.....0°	54	10.....9°	
	2.....1	48	20.....18	
	3.....2	42	30.....27	
	4.....3	36	40.....36	
	5.....4	30	50.....45	
	6.....5	24	60.....54	
	7.....6	18	70.....63	
	8.....7	12	80.....72	
	9.....8	6	90.....81	
	10.....9	0	100.....90	
Minutes.	1′.....0′	32″.4	10.....5′	24″
	2.....1	4.8	20.....10	48
	3.....1	37.2	30.....16	12
	4.....2	9.6	40.....21	36
	5.....2	42	50.....27	0
	6.....3	14.4	60.....32	24
	7.....3	46.8	70.....37	48
	8.....4	19.2	80.....43	12
	9.....4	51.6	90.....48	36
	10.....5	24	100.....54	0
Seconds.	1″.....0″.324		10.....3″.24	
	2.....0.648		20.....6.48	
	3.....0.972		30.....9.72	
	4.....1.296		40.....12.96	
	5.....1.62		50.....16.2	
	6.....1.944		60.....19.44	
	7.....2.268		70.....22.68	
	8.....2.592		80.....25.92	
	9.....2.916		90.....29.16	
	10.....3.24		100.....32.4	

CHAP. II.

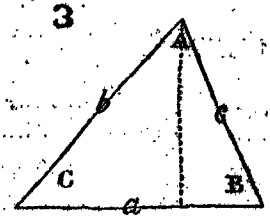
Expressions, for the Sines and Cosines of the Angles of a Triangle, in Terms of the Sides; for the Sine and Cosine of the Sum and Difference of two Arcs or Angles, &c.

THE first step in this investigation will be made by the solution of the following Problem:

PROBLEM 1. In an oblique-angled triangle, it is required to express the cosines of the angles in terms of the sides.

Let the 3 angles be A, B, C, the opposite sides, a, b, c.

Let the line between the vertex of the angle C and the point



where a perpendicular from the vertex of A on the line a cuts a be p; then, p is called the cosine of C to the radius b, and (by Art. 18) $= b \cdot \cos. C$, when the radius $= 1$.

Now, by *Euclid*, Book II, Prop. 12 and 13,

$$c^2 = a^2 + b^2 - 2ap$$
$$= a^2 + b^2 - 2ab \cdot \cos. C;$$

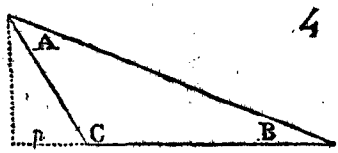
consequently, $\cos. C = \frac{a^2 + b^2 - c^2}{2ab}$.

If we investigate cos. B, and cos. A, the process will be exactly similar, and the result similar, that is

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From these expressions, the angles of a triangle may be found when the sides are given.



If C be obtuse, then $p = b \cdot \cos. (\pi - C)$

and $c^2 = a^2 + b^2 + 2ap$

$= a^2 + b^2 + 2ab \cdot \cos. (\pi - C)$

$= a^2 + b^2 - 2ab \cdot \cos. C$ { since $\cos. (\pi - C) = -\cos. C$ };

consequently, $\cos. C = \frac{a^2 + b^2 - c^2}{2ab}$, as before, or the expression is the same, whether C be less or greater than a right angle.

PROBLEM 2. Let it be required to express the sines of the angles in terms of the sides.

By *Euclid*, Book I. Prop. 47. $\sin.^2 A = (\text{rad}^s.)^2 - \cos.^2 A$ $= (\text{when rad}^s. = 1)\ 1 - \cos.^2 A = (1 + \cos. A)(1 - \cos. A)$; since, the difference of the squares of two quantities is equal to the product of their sum and difference; hence, we may find the value of $\sin.^2 A$, by finding, from the preceding Problem, $1 + \cos. A$, and $1 - \cos. A$, and then, by multiplying together those quantities.

Now, $1 - \cos. A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$.

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)(a + c - b)}{2bc}, \text{ (by l. 14, \&c.)}$$

similarly, * $1 + \cos. A = \frac{(b^2 + 2bc + c^2) - a^2}{2bc}$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(a+b+c)\ (b+c-a)}{2bc} \text{ (by p. 21. l. 14, \&c.)}$$

Hence multiplying, $1 + \cos. A$, and $1 - \cos. A$,

$$\sin.^2 A = \frac{(a+b+c)(b+c-a)(a+b-c)(a+c-b)}{4b^2c^2};$$

or, since $a+b-c=2\left(\frac{a+b+c}{2}-c\right)$, $b+c-a=2\left(\frac{a+b+c}{2}-a\right)$, &c. $\sin.^2 A =$

$$\frac{2.2.2.2}{4b^2c^2}\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)\right\} =$$

$$\frac{2^2}{b^2c^2}\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)\right\},$$

This is the expression for the $\sin.^2 A$, formed by means of that for $\cos. A$. But, the expressions for $\cos. B$, $\cos. C$ are precisely similar to that for $\cos. A$, and, therefore, the $\sin.^2 B$ and $\sin.^2 C$ formed from them, by the same process, must be expressed by similar fractions; in which fractions, the numerators must, from the nature of their composition, be the same as the numerator for $\sin.^2 A$, and the denominators will be, respectively, a^2c^2, a^2b^2. Let N^2 represent the numerator, then

$$\sin. A = \frac{N}{bc},\ \sin. B = \frac{N}{ac},\ \sin. C = \frac{N}{ab}.$$

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* $1 + \cos. A = \text{ver. sin. } (180° - A)$;

$\therefore$ ver. sin. $(180° - A) = \frac{(b+c)^2 - a^2}{2bc}$, or, differently expressed,

$4bc : (b+c)^2 - a^2 :: 2$ (the diameter) : ver. sin. $(180° - A)$ which is Halley's Theorem. *Phil. Trans.* No. 349. p. 466. Halley calls it "A new Theorem of good use in Trigonometry."

Cor. 1. Since sin. $B = \frac{N}{ac}$, and sin. $C = \frac{N}{ab}$, we have $\frac{\text{sin. } B}{\text{sin. } C} = \frac{b}{c}$, or if this equality be thrown into a proportion, sin. B : sin. C :: b : c. This relation, however, of the sines of the angles to the sides opposite, may be immediately deduced from (Fig. 3.); for the perpendicular (q) on a from the vertex of A, is the sine of B to the radius c, or $q = c$ sin. B (radius = 1); similarly q is the sine of C to the radius b, or, $q = b$ sin. C (rad. 1) $\therefore$ since $q = q$, c sin. $B = b$ sin. C.

Cor. 2. Hence the area of a triangle may be expressed in terms of its sides, for, (see Fig. 3) area $= \frac{a.\,perp^r.}{2} = \frac{ac}{2}$. sin. $B =$

$$\sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)\right\}}.$$

Problem 3. It is required to express the sine of the sum of two arcs, in terms of the sines and cosines of the simple arcs.

From the two preceding Problems,

$$\text{sin. } A = \frac{N}{bc}, \qquad \text{sin. } B = \frac{N}{ac}$$

$$\text{and, cos. } B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \text{cos. } A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Hence, sin. A.cos. B + cos. A.sin. $B = \frac{N(a^2+c^2-b^2+b^2+c^2-a^2)}{2abc^2}$

$$= \frac{2Nc^2}{2abc^2} = \frac{N}{ab}.$$

But sin. $C = \frac{N}{ab}$, and sin. C = sin. $\{\pi - (A+B)\}$ = sin. $(A+B)$, for $A + B$ is supplement to $\pi - (A + B)$ as $A + B + C = \pi$.

Hence sin. $(A+B)$ = sin. A . cos. B + cos. A . sin. B. [1]*

* It has been objected, and rightly, that this formula is not thus obtained by *direct* investigation, but discovered to be true by observation on the values of sin. A, cos. A, sin. B, cos. B, and by previous knowledge of those values. There is no difficulty in directly investigating it, by means of a construction (see Thomas Simpson's *Trig.* p. 54.)

An

This is the fundamental form, from which almost all other Trigonometrical forms may be deduced.

Cor. 1.

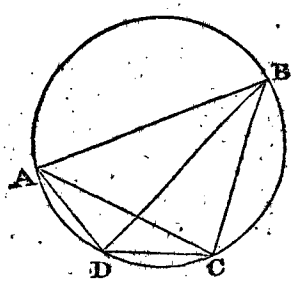
$\cos.(A+B) =$ (Art. 10. p. 6.) $\sin.\left(\frac{\pi}{2} - (A+B)\right) =$ (Art. 5.

An able critic, however, (see *Edin. Review*, No. XXXIII. p. 128.) suggests, that this, and similar formulæ, may be naturally and simply deduced by means of a Lemma of Ptolemy in his μεγαλη συνταξις, and which in Simpson's edition of *Euclid* is the Prop. *D* of the Sixth Book. Thus, according to that proposition,

$$AB \times CD + BC \times AD = AC \times BD.$$

Let the arc $AD = DC = 2B$, and the arc $BAD = 2A$; then,

$(AB + BC)\ AD = AC \times BD$, or, (see p. 12, l. 12, &c.)



$\{2 \cdot \sin.(A-B) + 2 \sin.(A+B)\}\ 2 \sin. B = 2 \sin. 2B \times 2 \sin. A$ (*a*)

Let $A = \frac{\pi}{2}$, then,

$$\left\{\sin.\left(\frac{\pi}{2} - B\right) + \sin.\left(\frac{\pi}{2} + B\right)\right\} \sin. B = \sin. 2B \times \sin. \frac{\pi}{2}$$

but, (p. 6.) $\sin.\left(\frac{\pi}{2} - B,\right)$ and $\sin.\left(\frac{\pi}{2} + B\right) = \cos. B$ and $\sin. \frac{\pi}{2} = 1$;

$\therefore 2 \cdot \cos. B \times \sin. B = \sin. 2B$ (the same as in p. 13.) substituting this in the form (*a*) l. 14.

$$\sin.(A-B) + \sin.(A+B) = 2 \sin. A \cdot \cos. B \quad (b).$$

Since in this formula A and B may be any arcs provided that $A > B$, substitute $\frac{\pi}{2} + A$ instead of A, and $\frac{\pi}{2} + B$ instead of B, then

sin.

p. 4.) sin. $\left(\frac{\pi}{2}+(A+B)\right)$

but, sin. $\left(\frac{\pi}{2}+(A+B)\right)$ = sin. $\left\{\left(\frac{\pi}{2}+A\right)+B\right\}$

= sin. $\left(\frac{\pi}{2}+A\right)$ cos. B + cos. $\left(\frac{\pi}{2}+A\right)$. sin. B,

which is derived from the form [1], by substituting, instead of A, $\left(\frac{\pi}{2}+A\right)$.

But by Art. 10. sin. $\left(\frac{\pi}{2}+A\right)$ = cos. A,

and cos. $\left(\frac{\pi}{2}+A\right)$ = − sin. A.

Hence, cos. $(A+B)$ = cos. A . cos. B − sin. A . sin. B.....[2]

Cor. 2. By page 6,

sin. $(A-B)$ = cos. $\left(\frac{\pi}{2}-(A-B)\right)$ = cos. $\left\{\left(\frac{\pi}{2}-A\right)+B\right\}$

But, by the formula [2] that has been just established,

---

sin. $(A-B)$ + sin. $(\pi+A+B)$ = 2.sin. $\left(\frac{\pi}{2}+A\right)$ cos. $\left(\frac{\pi}{2}+B\right)$

But, by Table, p. 9, and pp. 6, &c.

sin. $(\pi+A+B)$ = − sin. $(A+B)$; sin. $\left(\frac{\pi}{2}+A\right)$ = cos. A

and cos. $\left(\frac{\pi}{2}+B\right)$ = − sin. B; therefore

sin. $(A-B)$ − sin. $(A+B)$ = − 2 cos. A . sin. B (c).

Hence, adding and subtracting (b) and (c), there result

sin. $(A-B)$ = sin. A . cos. B − cos. A . sin. B,

and sin. $(A+B)$ = sin. A . cos. B + cos. A . sin. B,

and the deduction of other forms from these fundamental ones may be conducted as it is in the text.

$$\cos.\left\{\left(\frac{\pi}{2}-A\right)+B\right\} =$$

$$\cos.\left(\frac{\pi}{2}-A\right)\cos.B-\sin.\left(\frac{\pi}{2}-A\right)\sin.B$$

$$= \sin.A.\cos.B-\cos.A.\sin.B$$

(by p. 6.)

Hence, therefore,

$$\sin.(A-B) = \sin.A\cos.B-\cos.A\sin.B \ldots\ldots\ldots\ldots[3]$$

Again, by page 6,

$$\cos.(A-B)=\sin.\left(\frac{\pi}{2}-(A-B)\right)=\sin.\left\{\left(\frac{\pi}{2}-A\right)+B\right\}$$

But, by the formula [1]

$$\sin.\left\{\left(\frac{\pi}{2}-A\right)+B\right\}=\sin.\left(\frac{\pi}{2}-A\right)\cos.B+\cos.\left(\frac{\pi}{2}-A\right)\sin.B$$

$$= \cos.A\cos.B+\sin.A.\sin.B \text{ (by p. 6.)}$$

Hence, therefore,

$$\cos.(A-B) = \cos.A.\cos.B+\sin.A.\sin.B \ldots\ldots\ldots\ldots[4]$$

Cor. 3. Add together the forms [1] and [3,] and there results $\sin.(A+B)+\sin.(A-B)=2.\sin.A.\cos.B \ldots\ldots[a]$

Subtract [3] from [1], and

$$\sin.(A+B)-\sin.(A-B)=2.\cos.A.\sin.B \ldots\ldots[b]$$

Multiply [1] and [3],

and, the right hand side of the equation is =

$$\sin.^2A\times\cos.^2B-\cos.^2A\times\sin.^2B =$$

$$\sin.^2A(1-\sin.^2B)-(1-\sin.^2A)\sin.^2B =$$

$$\sin.^2A-\sin.^2B.$$

Hence,

$$\sin.(A+B)\times\sin.(A-B) = \sin.^2A-\sin.^2B \ldots\ldots[c]$$

Add [2] and [4], and

$$\cos.(A-B)+\cos.(A+B) = 2\cos.A.\cos.B \ldots\ldots[d]$$

Subtract [2] from [4], and

$$\cos.(A-B)-\cos.(A+B) = 2\sin.A.\sin.B \ldots\ldots[e]$$

Cor. 4. These last forms may be differently expressed,

$$\text{since, } A = \frac{A+B}{2} + \frac{A-B}{2} = \frac{S}{2} + \frac{D}{2}$$
$$B = \frac{A+B}{2} - \frac{A-B}{2} = \frac{S}{2} - \frac{D}{2}$$

making $S = A + B$, $D = A - B$

we have from [*a*]

$$\sin. S + \sin. D = 2 \sin. \left(\frac{S+D}{2}\right) . \cos. \left(\frac{S-D}{2}\right),$$

and from [*b*]

$$\sin. S - \sin. D = 2 \cos. \left(\frac{S+D}{2}\right) . \sin. \left(\frac{S-D}{2}\right),$$

or, since S and D are any arcs subject to this condition alone, that $S > D$, and since, in a series of formulæ, it is convenient to use the same characters, instead of S and D we may use A and B, and then,

$$\sin. A + \sin. B = 2 \sin. \left(\frac{A+B}{2}\right) . \cos. \left(\frac{A-B}{2}\right) \ldots\ldots\ldots [5]$$

$$\sin. A - \sin. B = 2 \cos. \left(\frac{A+B}{2}\right) . \sin. \left(\frac{A-B}{2}\right) \ldots\ldots\ldots [6]$$

By a similar process we may transform [*d*] and [*e*] into these,

$$\cos. A + \cos. B = 2 \cos. \left(\frac{A+B}{2}\right) . \cos. \left(\frac{A-B}{2}\right) \ldots\ldots\ldots [7]$$

$$\cos. B - \cos. A = 2 \sin. \left(\frac{A+B}{2}\right) . \sin. \left(\frac{A-B}{2}\right) \ldots\ldots\ldots [8]$$

Cor. 5. Divide [5] by [6], and

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin. \left(\frac{A+B}{2}\right)}{\cos. \left(\frac{A+B}{2}\right)} \cdot \frac{\cos. \left(\frac{A-B}{2}\right)}{\sin. \left(\frac{A-B}{2}\right)}$$

$$= \tan. \left(\frac{A+B}{2}\right) . \cot. \left(\frac{A-B}{2}\right),$$

$$\text{or, } = \frac{\tan. \left(\frac{A+B}{2}\right)}{\tan. \left(\frac{A-B}{2}\right)} \ldots\ldots\ldots\ldots [f]$$

similarly, $\frac{\cos. A+\cos. B}{\cos. B-\cos. A} = \frac{1}{\tan.\left(\frac{A+B}{2}\right).\tan.\left(\frac{A-B}{2}\right)}$ [g]

and, $\frac{\sin. A+\sin. B}{\cos. A+\cos. B} = \tan.\left(\frac{A+B}{2}\right)$ .....[h]

which, in a particular case, that is, when $B = 0$, becomes $\frac{\sin. A}{1+\cos. A} = \tan.\frac{A}{2}$, or $\frac{\sqrt{1-\cos.^2 A}}{1+\cos. A}$, or $\sqrt{\frac{1-\cos. A}{1+\cos. A}} = \tan.\frac{A}{2}$.

From this last expression, we may express, cos. A in terms of the tan. $\frac{A}{2}$, &c.; for since

$$\tan.^2\frac{A}{2} = \frac{1-\cos. A}{1+\cos. A}, \text{ we have}$$

$$\cos. A = \frac{1-\tan.^2\frac{A}{2}}{1+\tan.^2\frac{A}{2}}.$$

Cor. 6. The forms for the sine and cosine of $A \pm B$ being obtained, it is easy to deduce from them, the sine or cosine of any arc, such as $\frac{m\pi}{n} \pm A$, and nearly with as much convenience as by reference to the Table given in page 9, thus, by [1]

$$\sin.(\pi+A) = \sin.\pi.\cos. A+\cos.\pi.\sin. A,$$
$$\text{but } \sin.\pi = 0, \text{ and } \cos.\pi = -1;$$
$$\therefore \sin.(\pi+A) = -\sin. A.$$

Again, $\sin.\left(\frac{3\pi}{2}-A\right) = \sin.\frac{3\pi}{2}\cos. A - \cos.\frac{3\pi}{2}\sin. A$;

but, $\sin.\left(\frac{3\pi}{2}\right) = \sin.\left(\pi+\frac{\pi}{2}\right) = \sin.\pi.\cos.\frac{\pi}{2} + \cos.\pi.\sin.\frac{\pi}{2}$

$= -1$, since $\cos.\frac{\pi}{2} = 0$, and $\cos.\pi = -1$

$$\cos.\ \frac{3\pi}{2} = \cos.\left(\pi + \frac{\pi}{2}\right) = \cos.\ \pi.\ \cos.\ \frac{\pi}{2} - \sin.\ \pi\ \sin.\ \frac{\pi}{2} = 0.$$

The $\sin.\left(\frac{3\pi}{2} - A\right)$, therefore, $= -\cos.\ A$.

Again, $\cos.\left(\frac{3\pi}{2} - 3A\right) = \cos.\ \frac{3\pi}{2}\ \cos.\ 3A + \sin.\ \frac{3\pi}{2}\ \sin.\ 3A$

$= -\sin.\ 3A$, since $\cos.\ \frac{3\pi}{2} = 0$, and $\sin.\ \frac{3\pi}{2} = -1$.

Again, $\cos.\left(\frac{5\pi}{2} + A\right) = \cos.\ \frac{5\pi}{2}\ \cos.\ A - \sin.\ \frac{5\pi}{2}\ \sin.\ A$,

but $\cos.\ \frac{5\pi}{2} = \cos.\left(2\pi + \frac{\pi}{2}\right) = \cos.\ 2\pi.\ \cos.\ \frac{\pi}{2} - \sin.\ 2\pi.\ \sin.\ \frac{\pi}{2} = 0$

$\sin.\ \frac{5\pi}{2} = \sin.\left(2\pi + \frac{\pi}{2}\right) = \sin.\ 2\pi.\ \cos.\ \frac{\pi}{2} + \cos.\ 2.\ \pi.\ \sin.\ \frac{\pi}{2} = 1$;

$\therefore\ \cos.\left(\frac{5\pi}{2} + A\right) = -\sin.\ A$.

and, in like manner, other instances may be resolved.

We will now proceed to deduce expressions for the tangents of the sums and differences of arcs: and, the first step will be the solution of the following Problem, which, indeed, is little else than a corollary from the preceding results:

PROBLEM 4. It is required to express the tangent of the sum and difference of two arcs in terms of the tangents of the simple arcs.

Since (p. 10.) the tangent of an arc is equal to the sine divided by the cosine, if the arc be $A \pm B$

$$\tan.\ (A \pm B) = \frac{\sin.\ (A \pm B)}{\cos\ (A \pm)B} = \frac{\sin.\ A\ .\ \cos.\ B \pm \cos.\ A\ .\ \sin.\ B}{\cos.\ A\ .\ \cos.\ B \mp \sin.\ A.\ \sin.\ B}.$$

Now, since the object is to obtain an expression involving tan. A, tan. B, we must divide both numerator and denominator of the above fraction by cos. A. cos. B, an operation which will not change its real value; beginning then with the numerator

$$\frac{\sin.\ A\ .\ \cos.\ B \pm \cos.\ A\ .\ \sin.\ B}{\cos.\ A\ .\ \cos.\ B} = \frac{\sin.\ A}{\cos.\ A} \pm \frac{\sin.\ B}{\cos.\ B} = \tan.\ A \pm \tan.\ B$$

$$\frac{\cos. A \cos. B \mp \sin. A \sin. B}{\cos. A. \cos. B} = 1 \mp \frac{\sin. A \sin. B}{\cos. A \cos. B} = 1 \mp \tan. A. \tan. B$$

consequently, $\tan. (A \pm B) = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A. \tan. B}$ (rad. = 1) [9]

This formula may be used for determining the tangents of such arcs as $90° \pm A$, $180° \pm A$, &c. exactly, as in Cor. 6. p. 28, we shewed the formulæ for sin. $(A \pm B)$, &c. might be used in determining sin. $\left(\frac{m}{n}\pi \pm A\right)$: for instance,

$$\tan. (90° + A) = \frac{\tan. 90° + \tan. A}{1 - \tan. 90° \tan. A} = \frac{\infty + \tan. A}{1 - \infty \tan. A} = \frac{\infty}{-\infty \tan. A}$$

$$= -\frac{1}{\tan. A}: \text{ again, } \tan. (180° - A) = \frac{\tan. 180° - \tan. A}{1 + \tan. 180° \tan. A} = \frac{0 - \tan. A}{1 + 0 \tan. A}$$

$$= -\tan. A.$$

Cor. 2. Since tan. 45° (see p. 11.) = 1, if we make $A = 45°$, there will result from the preceding expression

$$\tan. (45° \pm B) = \frac{1 \pm \tan. B}{1 \mp \tan. B}.$$

Cor. 3. Let t, t', t'', &c. be the tangents of the arcs A, B C, &c. then by formula [9] considering $A + B$ as one arc,

$$\tan. (A + B + C) = \frac{\tan. (A + B) + \tan. C}{1 - \tan. (A + B) \tan. C} \qquad [q]$$

But, $\tan. (A + B) = \frac{t + t'}{1 - tt'}$;

∴ the numerator of the fraction [q] equals

$$\frac{t + t' + t'' - tt't''}{1 - tt'},$$

and the denominator, of the same fraction, equals

$$\frac{1 - (tt' + tt'' + t't'')}{1 - tt'}$$

Hence,

$$\tan. (A + B + C) = \frac{t + t' + t'' - tt't''}{1 - (tt' + tt'' + t't'')}.$$

If $A + B + C = \pi$, (which is the case when A, B, C, are the three angles of a triangle), since tan. $\pi = 0$,

$$t + t' + t'' - tt't'' = 0, \text{ or}$$
$$t + t' + t'' = tt't'',$$

which is the theorem given in the *Phil. Trans.* 1808. p. 122.

But, the theorem has an origin much more remote; for, the above formula for tan. $(A + B + C)$ and similar formulæ for the tangents of $A + B + C + D$, &c. were given as far back as the year, 1722, by John Bernoulli, and are inserted in the *Leipsick Acts* for that year, p. 361, and in the second volume of his Works, at p. 526.

The formulæ for the tangents of the sums of any arcs A, B, C, &c. are symmetrical in their composition, and their law is easily defined: suppose, the symbols $S_3\,(tt')$, $S_4\,(tt't'')$, &c. be made to represent, respectively,

$$tt' + tt'' + t't''$$
$$tt't'' + tt't''' + tt''t''' + t't''t''', \text{ &c.}$$

$$\text{then, tan. } (A + B + C) = \frac{S_3\,(t) - S_3\,(tt't'')}{1 - S_3\,(tt')}$$

$$\text{tan. } (A + B + C + D) = \frac{S_4\,(t) - S_4\,(tt't'')}{1 - S_4\,(tt') + S_4\,(tt't''t''')}$$

$$\text{tan. } (A + B + C + D + E) = \frac{S_5\,(t) - S_5\,(tt't'') + S_5\,(tt'\ldots t^{iv})}{1 - S_5\,(tt') + S_5\,(tt't''t''')},$$

&c.

These formulæ are easily shewn to be true on the principle, that, if the formula for the tangent of n arcs be true, that for $(n + 1)$ arcs must be true also: the latter inference being made by means of the form [9] p. 29.

If, instead of a radius $= 1$, we would introduce a radius $= r$ into the preceding formulæ, we must avail ourselves of the rule laid down in p. 15. Thus the formula [1] p. 23, becomes

$$r \cdot \text{sin. } (A + B) = \text{sin. } A \times \text{cos. } B + \text{cos. } A \times \text{sin. } B$$

the formula [c] p. 26, is the same whether the radius be 1 or r.

The formula $\tan.^2 \frac{A}{2} = \frac{1 - \cos. A}{1 + \cos. A}$ to radius $= 1$

becomes $\tan.^2 \frac{A}{2} = \frac{r^2 - r^2 . \cos. A}{r + \cos. A}$ when radius $= r$.

The formula [*h*] becomes

$$r \times \frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \left(\frac{A+B}{2}\right)$$

but, the formula [*f*], p. 27, remains the same, whether the radius be equal 1 or r.

We will now subjoin a few additional formulæ for the sines and cosines, &c. of the sums and differences of arcs, the investigation of which the Student, by pursuing a track similar to what has been already adopted, will easily discover,

$$\sin. (45° \pm A) = \frac{\cos. A \pm \sin. A}{\sqrt{2}}$$

$$\cos. (45° \mp A) = \frac{\cos. A \pm \sin. A}{\sqrt{2}}$$

$$\sin. (60° + A) - \sin. (60° - A) = \sin. A$$
$$\cos. (60° + A) + \cos. (60° - A) = \cos. A$$

$$\tan. (45° + A) = \frac{1 + \tan. A}{1 - \tan. A}$$

$$\tan. (45° - A) = \frac{1 - \tan. A}{1 + \tan. A}$$

$$\tan.^2 A - \tan.^2 B = \frac{\sin. (A-B) \sin. (A+B)}{\cos.^2 A \times \cos.^2 B}$$

$$\cot.^2 B - \cot.^2 A = \frac{\sin. (A-B) \sin. (A+B)}{\sin.^2 A \times \sin.^2 B}.$$

M. Cagnoli, in his *Trigonometry*, has collected into a Table, under one view, and for the purpose of reference, formulæ similar to the preceding. He has also in another Table (which is subjoined) exhibited the various values for the sine, cosine, and tangent of the angle *A*.

Values of sin. A.

1. $\cos. A \tan. A.$
2. $\dfrac{\cos. A}{\cot. A}.$
3. $\sqrt{(1 - \cos.^2 A)}.$
4. $\dfrac{1}{\sqrt{(1 + \cot.^2 A)}}.$
5. $\dfrac{\tan. A}{\sqrt{(1 + \tan.^2 A)}}.$
6. $2 \sin. \dfrac{A}{2} . \cos. \dfrac{A}{2}.$
7. $\sqrt{\left(\dfrac{1 - \cos. 2A}{2}\right)}.$
8. $\dfrac{2 \tan. \frac{A}{2}}{1 + \tan.^2 \frac{A}{2}}$
9. $\dfrac{2}{\cot. \frac{A}{2} + \tan. \frac{A}{2}}$
10. $\dfrac{\sin. (30° + A) - \sin. (30° - A)}{\sqrt{3}}$
11. $2 \sin.^2 \left(45° + \dfrac{A}{2}\right) - 1.$
12. $1 - 2 \sin.^2 \left(45° - \dfrac{A}{2}\right)$
13. $\dfrac{1 - \tan.^2 \left(45° - \frac{A}{2}\right)}{1 + \tan.^2 \left(45° - \frac{A}{2}\right)}$
14. $\dfrac{\tan. \left(45° + \frac{A}{2}\right) - \tan. \left(45° - \frac{A}{2}\right)}{\tan. \left(45° + \frac{A}{2}\right) + \tan. \left(45° - \frac{A}{2}\right)}$
15. $\sin. (60° + A) - \sin. (60° - A).$

Values of the cos. A.

1. $\dfrac{\sin. A}{\tan. A}.$
2. $\sin. A \cot. A.$
3. $\sqrt{(1 - \sin.^2 A)}.$
4. $\dfrac{1}{\sqrt{(1 + \tan.^2 A)}}.$
5. $\dfrac{\cot. A}{\sqrt{(1 + \cot.^2 A)}}.$
6. $\cos.^2 \dfrac{A}{2} - \sin.^2 \dfrac{A}{2}.$
7. $1 - 2 \sin.^2 \dfrac{A}{2}.$
8. $2 . \cos.^2 \dfrac{A}{2} - 1.$
9. $\sqrt{\left(\dfrac{1 + \cos. 2A}{2}\right)}.$
10. $\dfrac{1 - \tan.^2 \frac{A}{2}}{1 + \tan.^2 \frac{A}{2}}$
11. $\dfrac{\cot. \frac{A}{2} - \tan. \frac{A}{2}}{\cot. \frac{A}{2} + \tan. \frac{A}{2}}.$
12. $\dfrac{1}{1 + \tan. A \tan. \frac{A}{2}}.$
13. $\dfrac{2}{\tan. \left(45° + \frac{A}{2}\right) + \cot. \left(45° + \frac{A}{2}\right)}$
14. $2 \cos. \left(45° + \dfrac{A}{2}\right) \cos. \left(45° - \dfrac{A}{2}\right)$
15. $\cos. (60° + A) + \cos. (60° - A).$

Values of tan. A.

1. $\dfrac{\sin. A}{\cos. A}$

2. $\dfrac{1}{\cot. A}$

3. $\sqrt{\left(\dfrac{1}{\cos.^2 A} - 1\right)}$

4. $\dfrac{\sin. A}{\sqrt{(1 - \sin.^2 A)}}$

5. $\dfrac{\sqrt{(1 - \cos.^2 A)}}{\cos. A}$

6. $\dfrac{2 \tan. \frac{A}{2}}{1 - \tan.^2 \frac{A}{2}}$

7. $\dfrac{2 \cot. \frac{A}{2}}{\cot.^2 \frac{A}{2} - 1}$

8. $\dfrac{2}{\cot. \frac{A}{2} - \tan. \frac{A}{2}}$

9. $\cot. A - 2 \cot. 2A$

10. $\dfrac{1 - \cos. 2A}{\sin. 2A}$

11. $\dfrac{\sin. 2A}{1 + \cos. 2A}$

12. $\sqrt{\left(\dfrac{1 - \cos. 2A}{1 + \cos. 2A}\right)}$

13. $\dfrac{\tan. \left(45° + \frac{A}{2}\right) - \tan. \left(45° - \frac{A}{2}\right)}{2}$.

The investigation of some of these expressions has been already given; and, by pursuing its plan, the Student will, without difficulty, be able to accomplish that of the others.

But, the Student, whose object is utility, will feel averse from their investigation, should he suspect them to be mere Trigonometrical curiosities. Such however is not their character; on the contrary, they, in many instances, materially expedite calculation, and furnish to the general language of analysis convenient forms and modes of expression. It is, in accomplishing this latter purpose, that Trigonometrical formulæ are chiefly useful: they serve to conduct investigation where the object has no concern whatever with the properties of triangles.

Yet, the investigation of the properties of triangles was the object for which Trigonometry was originally invented; and, if the Student purposes to limit his enquiries to that object alone, he need not, in quest of the requisite formulæ, advance farther than the present Chapter. He may immediately pass on to the fifth Chapter and apply what he has already learned. If, however, his views should extend farther, and he should wish to be possessed of Trigonometry and its formulæ as instruments of language, he must pursue his researches, become conversant with expressions merely analytical, and, for a time, defer their application.

In order that this latter plan may be adopted, we will, in the next Chapter, continue the deduction of Trigonometrical formulæ.

CHAP. III.

On the Sines, Cosines, &c. of multiple Arcs.—Powers of the Sine and Cosine of the simple Arc.—Series of the Cosines of Arcs in Arithmetical Progression.—Vieta's, Waring's, and Cotes's Properties of Curves.—De Moivre's Expression for the Sine and Cosine of a multiple Arc by means of imaginary Symbols.

PROBLEM 5. IT is required to express the sine and cosine of twice an arc, in terms of the sine and cosine of the simple arc.

By form [1], page 23,

$$\text{sin. } (A+B) = \text{sin. } A. \text{ cos. } B + \text{cos. } A. \text{ sin. } B.$$

Let $B = A$;

$\therefore$ sin. $(2A) =$ sin. $A.$ cos. $A +$ cos. $A.$ sin. $A = 2$ sin. A cos. A;
or, by the Rule of page 15, to rad^s. r, r sin. $2A = 2$ sin. $A.$ cos. A *

Again, by the form [2] of page 25,

$$\text{cos. } (A + B) = \text{cos. } A. \text{ cos. } B - \text{sin. } A. \text{ sin. } B.$$

Let $A = B$; $\therefore$ cos. $2A =$ cos.^{2} $A -$ sin.^{2} A;

or, $=$ cos.^{2} $A - (1 -$ cos.^{2} $A) = 2.$ cos.^{2} $A - 1$;

or, $= 1 -$ sin.^{2} $A -$ sin.^{2} $A = 1 - 2$ sin.^{2} A.

If we employ a radius $= r$, then, by the Rule of p. 15,

$r.$ cos. $2A = 2$ cos.^{2} $A - r^2$, or $= r^2 - 2.$ sin.^{2} A.

COR. 1. By transposition,

$r^2 - r.$ cos. $2A$, or $r(r -$ cos. $2A) = 2$ sin.^{2} A.

* This result has been (p. 13.) already obtained, but it is here repeated, since it is the first of a series of formulæ deduced on the same principle.

Now, $r - \cos. 2A = \text{ver. sin. } 2A$, consequently,

$$r \,.\, (\text{ver. sin. } 2A) = 2 \sin.^2 A \text{ (see p. 13.)}$$

which equality, thrown into a proportion, is

$$\text{ver. sin. } 2A : \sin. A :: \sin. A : \frac{r}{2} ;$$

or, expressed in general terms, announces, that the sine of an angle is a mean proportional between the versed sine of twice the same angle and the semi-radius.

$$\text{Again, } r^2 + r \,.\, \cos. 2A = 2 \cos.^2 A$$
$$\text{or } r(r + \cos. 2A) = 2 \cos.^2 A,$$
$$\text{or } r(\text{ver. sin. supp}^{t}. \text{ of } 2A) = 2 \cos.^2 A ;$$

or, if we call the ver. sin. (supplement of arc), the suversed sine $r \,.\, \text{suversin. } (2A) = 2 \cos.^2 A$, which equality, like the preceding, may be expressed under the form of a proportion, or in general terms.

Cor. 2. Hence the sine of $30° = \frac{1}{2}$, radius $= 1$,

For, $\sin. 60° = \sin. (2 \,.\, 30°) = 2 \sin. 30° . \cos. 30°$ [Prob. 5.]
but, $\cos. 30° = \sin. (90° - 30°) = \sin. 60°$;
$\therefore \sin. 60° = 2 \sin. 30° . \sin. 60°$;

or $\sin. 30° = \frac{1}{2}$, and consequently $\sin. 60°$, or $\cos. 30° = \frac{\sqrt{3}}{2}$.

Cor. 3. Since $\cos. 2A = 1 - 2 \sin.^2 A$, or $= 2 \cos.^2 A - 1$

$$\sin. A = \sqrt{\tfrac{1}{2}} \sqrt{(1 - \cos. 2A)}$$
$$\cos. A = \sqrt{\tfrac{1}{2}} \sqrt{(1 + \cos. 2A)}.$$

Let $x - y = \sqrt{(1 - \cos. 2A)}$, or $x^2 - 2xy + y^2 = 1 - \cos. 2A$.

Assume $x^2 + y^2 = 1$

then, $2xy = \cos. 2A$,

hence, eliminating y

$$x^2 + \frac{\cos.^2 2A}{4x^2} = 1,$$

and $x^4 - x^2 + \frac{1}{4} = \frac{1}{4} \left\{ 1 - \cos.^2 2A \right\} = \frac{1}{4} \sin.^2 2A$;

$$\therefore x^2 = \tfrac{1}{2} \pm \frac{\sin. 2A}{2}, \text{ and } x = \sqrt{\tfrac{1}{2}} \sqrt{(1 \pm \sin. 2A)},$$

and $y^2 = \frac{1}{2} \mp \frac{\sin. 2A}{2}$, and $y = \sqrt{\frac{1}{2}}\sqrt{(1 \mp \sin. 2A)}$.

Hence

$$x - y = \sqrt{\frac{1}{2}}\sqrt{(1 + \sin. 2A)} - \sqrt{\frac{1}{2}}\sqrt{(1 - \sin. 2A)},$$

but $x - y = \sqrt{(1 - \cos. 2A)}$, and $\sin. A = \sqrt{\frac{1}{2}}\sqrt{(1 - \cos. 2A)}$.

The sin. A and cos. A may therefore be thus expressed,

$$\sin. A = \tfrac{1}{2}\sqrt{(1 + \sin. 2A)} - \tfrac{1}{2}\sqrt{(1 - \sin. 2A)}$$
$$\cos. A = \tfrac{1}{2}\sqrt{(1 + \sin. 2A)} + \tfrac{1}{2}\sqrt{(1 - \sin. 2A)}*.$$

These two formulæ are useful either to compute arithmetically the sines, cosines, &c. of arcs, or to examine their accuracy when computed by other formulæ; and, performing the latter office, they are called *Formulæ of Verification*. It is easy to perceive their use in computing sines, cosines, &c.; for if we take sin. $2A$, a known quantity, for instance, the sin. 30° which equals $\frac{1}{2}$, we may, by successive substitutions, regularly deduce the sines of 15°, 7° 30′, &c. Thus,

$$\sin. 15° = \tfrac{1}{2}\sqrt{(1 + \tfrac{1}{2})} - \tfrac{1}{2}\sqrt{(1 - \tfrac{1}{2})} = .258819$$
$$\sin. 7° 30' = \tfrac{1}{2}\sqrt{(1.258819)} - \tfrac{1}{2}\sqrt{(.741181)} = .1305262$$
$$\sin. 3° 45' =, \&c.$$

PROBLEM 6. It is required to express the sine and cosine of 3 times an arc, 4 times an arc, &c. in terms of the sine and cosine of the simple arc.

If we substitute in the formula for cos. $(A + B)$ (p. 25.) $2A$ instead of B, we have

$$\cos. (A + 2A) = \cos. 2A . \cos. A - \sin. 2A . \sin. A;$$

but, by the preceding Problem,

$$\sin. 2A = 2 \sin. A \cos. A,$$
$$\text{and } \cos. 2A = 2 . \cos.^2 A - 1;$$

* From these expressions, the sides of octahedrons, dodecahedrons, &c. may easily be deduced from those of squares, hexagons, &c.

$$\therefore \cos. 3A = (2 . \cos.^2 A - 1) \cos. A - 2 \cos. A . \sin.^2 A$$
$$= 2 . \cos.^3 A - \cos. A - 2 \cos. A (1 - \cos.^2 A)$$
$$= 4 \cos.^3 A - 3 \cos. A, \text{ when the radius} = 1,$$

and $r^2 . \cos. 3A = 4 \cos.^3 A - 3r^2 . \cos. A$, when the radius $= r$.

This form, if we substitute therein, instead of the arc $3A$, $\frac{3\pi}{2} - 3A$ gives us

$$\cos. \left(\frac{3\pi}{2} - 3A\right) = 4 . \left(\cos. \frac{\pi}{2} - A\right)^3 - 3 . \cos. \left(\frac{\pi}{2} - A\right)$$

but, by Cor. 6, Prob. 3, page 29,

$$\cos. \left(\frac{3\pi}{2} - 3A\right) = -\sin. 3A, \cos. \left(\frac{\pi}{2} - A\right) = \sin. A;$$

consequently,

$$\sin. 3A = 3 \sin. A - 4 . \sin.^3 A, \text{ when the radius} = 1,$$

and $r^2 . \sin. 3A = 3r^2 . \sin. A - 4 \sin.^3 A$, when the radius $= r$.

By a similar method may the $\cos. (4A) = \cos. (3A + A)$ or $= \cos. (2A + 2A)$, and the $\cos. (5A) = \cos. (4A + A)$ or $= \cos. (3A + 2A)$, &c. be deduced. But, the successive formation of the cosines and sines of multiple arcs may, most easily, be effected after the following manner:

By the form [*d*], page 26,

$$\cos. (A + B) + \cos. (A - B) = 2 \cos. A . \cos. B \ldots\ldots A > B$$

or, $\cos. (B + A) + \cos. (B - A) = 2 \cos. B . \cos. A \ldots\ldots B > A.$

Let $B = nA$, then transposing

$$\cos. (n + 1) A = 2 . \cos. nA . \cos. A - \cos. (n - 1) A,$$

and hence from $\cos. (n - 1) A$, and $\cos. nA$, may be assigned $\cos. (n + 1) A$: for instance,

$$\text{if } n = 1, \cos. (n - 1) A = \cos. 0 = 1;$$
$$\therefore \cos. 2A = 2 \cos.^2 A - 1 \ldots\ldots\ldots\ldots [c^{II}]$$

If $n = 2$,

$$\cos. 3A = 2 . \cos. 2A . \cos. A - \cos. A = 4 . \cos.^3 A - 3 \cos. A \; [c^{III}]$$

If $n = 3$,

$$\cos. 4A = 2 \cos. 3A . \cos. A - \cos. 2A$$
$$= 2(4 \cos.^3 A - 3 \cos. A) \cos. A - (2 \cos.^2 A - 1)$$
$$= 8 \cos.^4 A - 8 \cos.^2 A + 1 \ldots\ldots\ldots [c^{IV}],$$

And, by a like process, if $n = 4$,

$$\cos. 5A =$$

$$16\cos.^5 A - 16\cos.^3 A + 2\cos. A - (4\cos.^3 A - 3\cos. A)$$

$$= 16\cos.^5 A - 20\cos.^3 A + 5\cos. A \ldots\ldots\ldots\ldots [c^V]$$

$$\text{or, } 2\cos. 5A = (2\cos. A)^5 - 5(2\cos. A)^3 + 10\cos. A$$

Similarly, if $[n = 5]$

$$\cos. 6A = 32\cos.^6 A - 40\cos.^4 A + 10\cos.^2 A$$
$$- (8\cos.^4 A - 8\cos.^2 A + 1)$$
$$= 32\cos.^6 A - 48\cos.^4 A + 18\cos.^2 A - 1 \ldots\ldots [c^{VI}]$$

$$\text{or, } 2\,.\cos. 6A = 2^6\,.\cos.^6 A - 6\,.\,2^4\cos.^4 A + \frac{6\,.\,3}{2}2^2\cos.^2 A - 2$$

and the general form is

$$2\cos. mA = (2\cos. A)^m - m(2\cos. A)^{m-2} + \frac{m\,.\,(m-3)}{2}\,.\,(2\cos. A)^{m-4} \qquad [c^M]$$

$$- \frac{m\,.\,(m-4)\,(m-5)}{2\,.\,3}\,(2\cos. A)^{m-6} + \&c.$$

The formula for the sines of multiple arcs may be deduced from those of the cosines, and, on the same principle as that which has been already used in deducing sin. $3A$; by substituting, in the form for cos. $5A$, $\frac{5\pi}{2} - 5A$, instead of $5A$, we have

$$\cos. \left(\frac{5\pi}{2} - 5A\right) =$$

$$16\left(\cos. \frac{\pi}{2} - A\right)^5 - 20\left(\cos. \frac{\pi}{2} - A\right)^3 + 5\,.\cos. \left(\frac{\pi}{2} - A\right)$$

but by Cor. 6, Prob. 3, $\cos. \left(\frac{5\pi}{2} - 5A\right) = \sin. 5A$

$$\cos. \left(\frac{\pi}{2} - A\right) = \sin. A$$

consequently, $\sin. 5A = 16\sin.^5 A - 20\sin.^3 A + 5\sin. A.$

Or, the sines of multiple arcs may be successively deduced as the cosines have, on the same principle, and by like formulæ; thus, by the form $[a]$, p. 26.

$$\sin.(B+A)+\sin.(B-A)=2\sin.B.\cos.A.$$

Let $B=nA$, then, transposing

$$\sin.(n+1)A=2\sin.nA.\cos.A-\sin.(n-1)A,$$

hence, if $n=2$, $\sin.3A=2.\sin.2A.\cos.A-\sin.A$

$(\sin.2A=2\sin.A.\cos.A)=4\sin.A.\cos.^2A-\sin.A$.....$[s^{II}]$

$=3\sin.A-4\sin.^3A$..... $[s^{III}]$

Similarly, if $n=3$,

$\sin.4A=2\cos.A(3\sin.A-4\sin.^3A)-\sin.2A,$

$=6\sin.A.\cos.A-8\cos.A.\sin.^3A-2\sin.A.\cos.A$

$=(4\sin.A-8\sin.^3A)\cos.A$

or, $=(8\cos.^3A-4\cos.A)\sin.A$.....$[s^{IV}]$

Similarly, if $n=4$, $\sin.5A$

$=(4\sin.A-8\sin.^3A)\,2\cos.^2A-(3\sin.A-4\sin.^3A)$

$=8\sin.A-16\sin.^3A-8\sin.^3A+16\sin.^5A-(3\sin.A-4\sin.^3A)$

$=5\sin.A-20\sin.^3A+16\sin.^5A$.....$[s^{V}]$

or, $2\sin.5A=5.2\sin.A-5.(2\sin.A)^3+(2\sin.A)^5$.

The general expression for $\sin.mA$ [m odd] is

$$\sin.mA=$$

$$m.\sin.A-\frac{m.(m^2-1)}{2.3}\sin.^3A+\frac{m(m^2-1)(m^2-9)}{2.3.4.5}\sin.^5A-\&c.$$

and [m even] is

$$*\sin.mA=\cos.A\left\{m.\sin.A-\frac{m.(m^2-4)}{2.3}\sin.^3A+\right.$$

$$\left.\frac{m.(m^2-4)(m^2-16)}{2.3.4.5}\sin.^5A-\&c.\right\}$$

The sine and cosine of the multiple arc (mA) have been expressed in powers of the sine and cosine of the simple arc (A): but, if we express the cosine of the arc (A) by a particular binomial, the cosines of the multiple arcs admit of an expression rather remarkable; or, in other words, they may be said to possess a curious property: thus, let

$$2.\cos.A=x+\frac{1}{x};$$

---

* For the general demonstration of these forms, see the Appendix.

then, $2 \cos. 2A = 2(2 \cos.^2 A - 1) = 2\left\{\frac{1}{2}\left(x + \frac{1}{x}\right)^2 - 1\right\}$

$$= x^2 + \frac{1}{x^2}.$$

By the form, p. 39 l. 28,

$$2 . (\cos. 3A) = 4 \cos. 2A . \cos. A - 2 \cos. A$$

$$= \left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) - \left(x + \frac{1}{x}\right)$$

$$= x^3 + \frac{1}{x^3}.$$

Generally, if $2 . \cos. (n-1)A = x^{n-1} + \frac{1}{x^{n-1}}$,

and, $2 . \cos. nA = x^n + \frac{1}{x^n}$, then, since (p. 39,)

$$\cos. (n+1)A = 2 . \cos. nA . \cos. A - \cos. (n-1)A,$$

$$2 \cos. (n+1)A = \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)$$

$$* = x^{n+1} + \frac{1}{x^{n+1}}.$$

Hence, if the form were true for two successive inferior numbers, $n-1$, and n, it would be true for $n+1$; but, it has been proved to be true in those cases, when $n - 1 = 2$, and $n = 3$: consequently, it is true for $n + 1 = 4$, and so on successively for all whole superior numbers.

The above expression holds good also for $\cos. \frac{n+1}{2}A$, when n is an even number. For, since

$$\cos.^2 \frac{A}{2} = \frac{\cos. A + 1}{2},$$

$$4 . \cos.^2 \frac{A}{2} = 2 . \cos. A + 2 = x + 2 + \frac{1}{x};$$

---

* This mode of denoting the cosines of multiple arcs, which leads to several curious results, occurs first in De Moivre's *Miscellanea Analytica*, pages 8 and 16.

$$\therefore 2 . \cos. \frac{A}{2} = \sqrt{x} + \frac{1}{\sqrt{x}}.$$

Now

$$\cos. \frac{n+1}{2} A = 2 . \cos. \frac{n}{2} A \times \cos. \frac{A}{2} - \cos. \frac{n-1}{2} A$$

$$\therefore 2 . \cos. \frac{n+1}{2} A = \left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) - \left(x^{\frac{n-1}{2}} + \frac{1}{x^{\frac{n-1}{2}}}\right)$$

$$= x^{\frac{n+1}{2}} + \frac{1}{x^{\frac{n+1}{2}}};$$

therefore, as before, if the expression be true for $\cos. \frac{n-1}{2} A$, it is (since it certainly is so for $\cos. \frac{n}{2} A$, n being an even number) true, for $\cos. \frac{n+1}{2} . A$. But, (l. 1,) it is, when $n = 2$, true for $\cos. \frac{A}{2}$; therefore, it is true for $\cos. \frac{3A}{2}$; therefore for $\cos. \frac{5A}{2}$, and, by virtue of these successive inferences, generally true.

The above mode of expressing the cosines of multiple arcs is useful on several occasions: for instance, in finding the sum of a series, such as

$$\cos. A + \cos. 2A + \cos. 3A + \&c. + \cos. nA,$$

for, retaining the former notation, the sum is equal the sum of the two following series,

$$\frac{1}{2}(x + x^2 + x^3 + \&c. + x^n),$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c. + \frac{1}{x^n}\right);$$

$$\therefore = \frac{1}{2}\left(\frac{x^{n+1} - x}{x - 1} + \frac{x^n - 1}{x^n (x-1)}\right)$$

$$= \frac{1}{2}\left(\frac{(x^n - 1)(x^{n+1} + 1)}{x^n (x-1)}\right) = \frac{1}{2}\left\{\frac{\left(x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}\right)\left(x^{\frac{n+1}{2}} + \frac{1}{x^{\frac{n+1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}}\right\}$$

Now, $\dfrac{x^{\frac{n}{2}} - \dfrac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \dfrac{1}{\sqrt{x}}} = \sqrt{\left(\dfrac{x^{\frac{n}{2}} - \dfrac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \dfrac{1}{\sqrt{x}}}\right)^2} =$

$$\sqrt{\left(\frac{\left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right)^2 - 4}{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 4}\right)} = \sqrt{\left\{\frac{\left(2 \cdot \cos. \frac{nA}{2}\right)^2 - 4}{\left(2 \cdot \cos. \frac{A}{2}\right)^2 - 4}\right\}}$$

$$= \sqrt{\left(\frac{1 - \cos.^2 \frac{nA}{2}}{1 - \cos.^2 \frac{A}{2}}\right)} = \frac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}} : \text{consequently the sum}$$

of the series is equal $\dfrac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}} \times \cos. \left(\frac{n+1}{2}\right) A.$

Since the $\sin. nA = \dfrac{1}{2\sqrt{-1}} \left(x^n - \dfrac{1}{x^n}\right)$, we cannot, using the above notation, directly investigate the sum of a series such as

$$\sin. A + \sin. 2A + \sin. 3A + \&c.$$

except by introducing imaginary symbols. The investigation, however, is not difficult, and may conveniently be effected, (although by something like an artifice of calculation) in the following manner:

In the form [8] p. 27, if we substitute, instead of B and A, successively the arcs $\frac{A}{2}$ and $\frac{3A}{2}$, $\frac{3A}{2}$ and $\frac{5A}{2}$, &c. there will result

$$\cos. \frac{A}{2} - \cos. \frac{3A}{2} = 2 \cdot \sin. A, \sin. \frac{A}{2},$$

$$\cos. \frac{3A}{2} - \cos. \frac{5A}{2} = 2 \sin. 2A \cdot \sin. \frac{A}{2},$$

$$\cos. \frac{5A}{2} - \cos. \frac{7A}{2} = 2 \sin. 3A \cdot \sin. \frac{A}{2},$$

&c. &c.

Hence, by addition,

$$\cos.\frac{A}{2} - \cos.\frac{2n+1}{2}A = 2\sin.\frac{A}{2}(\sin.A + \sin.2A + \sin.3A + \&c. + \sin.nA);$$

consequently, $\sin.A + \sin.2A + \sin.3A + \&c. =$

$$\frac{\cos.\frac{A}{2} - \cos.\frac{2n+1}{2}A}{2\sin.\frac{A}{2}} = \frac{\sin.\frac{n}{2}.A \times \sin.\frac{n+1}{2}A}{\sin.\frac{A}{2}}.$$

By a similar artifice may the sum of the series

$$\frac{1}{2}\tan.\frac{A}{2} + \frac{1}{4}\tan.\frac{A}{4} + \frac{1}{8}\tan.\frac{A}{8} + \&c.$$

be found.

For, if in the expression for $\tan.(A+B)$ we make $A = B$, $\tan.2A = \frac{2\tan.A}{1-\tan.^2 A}$, and therefore (p. 11,)

$$\cot.2A = \frac{1-\tan.^2 A}{2\tan.A} = \frac{1}{2\tan.A} - \frac{\tan.A}{2} =$$

$\frac{1}{2}\cot.A - \frac{1}{2}\tan.A$ (which agrees with the 9th expression for $\tan.A$ in the Table, p. 34).

Hence, transposing, and substituting, instead of $2A$, A, successively the arcs, A and $\frac{A}{2}$, $\frac{A}{2}$ and $\frac{A}{4}$, &c. there results (continually dividing each successive formula by 2),

$$\frac{1}{2}\cot.\frac{A}{2} - \cot.A = \frac{1}{2}\tan.\frac{A}{2},$$

$$\frac{1}{4}\cot.\frac{A}{4} - \frac{1}{2}\cot.\frac{A}{2} = \frac{1}{4}\tan.\frac{A}{4},$$

$$\frac{1}{8}\cot.\frac{A}{8} - \frac{1}{4}\cot.\frac{A}{4} = \frac{1}{8}\tan.\frac{A}{8}, \&c.$$

Hence, by addition

$$\frac{1}{2^n}\cot.\frac{A}{2^n} - \cot.A = \frac{1}{2}\tan.\frac{A}{2} + \frac{1}{4}\tan.\frac{A}{4} + \frac{1}{8}\tan.\frac{A}{8} + \&c.$$

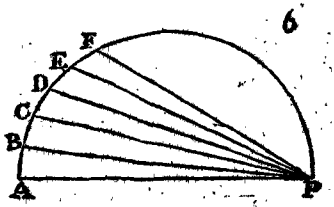
By a similar process,

$$\text{co-sec. } A + \text{co-sec. } 2A + \text{co-sec. } 4A + \&c. = \text{cot. } \frac{A}{2} - \text{cot. } 2^{n-1}A.$$

See Cagnoli's *Trigonometry*, second edit. p. 122.

We will now return to the expression $2\cos. nA = x^n + \frac{1}{x^n}$, and shew its use in demonstrating a property of curves given by Waring, in his *Proprietates Curvarum*, p. 110. The property is this:

If in a circle $ABCD$, &c. the radius of which is 1, equal arcs AB, BC, CD, &c. be taken, and if PB be taken $= p$, p being



the coefficient of the second term of the quadratic $Z^2 - pZ + 1 = 0$, of which the roots are, α, β;

then, $PB = \alpha + \beta$, or $= \alpha + \frac{1}{\alpha}$ since $\alpha\beta = 1$,

$PC = \alpha^2 + \beta^2$, or $= \alpha^2 + \frac{1}{\alpha^2}$,

$PD = \alpha^3 + \beta^3$, or $= \alpha^3 + \frac{1}{\alpha^3}$,

&c. = &c. or = &c.

now $PB = \text{chord } (\pi - AB) = 2 \text{ sin. } \left(\frac{\pi}{2} - \frac{AB}{2}\right) = 2. \text{ cos. } \frac{AB}{2}$

similarly, $PC = \text{chord } (\pi - AC) = 2 \text{ sin. } \left(\frac{\pi}{2} - \frac{AC}{2}\right) = 2. \text{ cos. } \frac{AC}{2}$

$PD = \ldots\ldots\ldots\ldots = 2 \text{ cos. } \frac{AD}{2}$

$PE =$ &c.

and it has been already proved, that if, (putting $\frac{AB}{2}$ for A)

$$2 \cdot \cos. \frac{AB}{2} = x + \frac{1}{x};$$

then $2 \cdot \cos. 2\left(\frac{AB}{2}\right)$, or $2 \cos. \left(\frac{AC}{2}\right) = x^2 + \frac{1}{x^2}$;

$$2 \cdot \cos. 3\left(\frac{AB}{2}\right), \text{ or } 2 \cdot \cos. \frac{AD}{2} = x^3 + \frac{1}{x^3}, \text{ \&c.}$$

hence, PB being $x + \frac{1}{x}$, or $\alpha + \frac{1}{\alpha}$, or, $\alpha + \beta$,

$$PC = \alpha^2 + \beta^2,$$
$$PD = \alpha^3 + \beta^3,$$
$$\text{\&c.} = \text{\&c.}$$

Vieta, p. 295, *Opera Mathematica*, Leyden, 1646, expresses the values of these chords, not by the sums of the powers of the roots, but by expressions equivalent to such sums: thus, he puts for PB, $1N$, or N; $(N)^2$ he represents by $1Q$, $(N)^3$ by $1C$, $(N)^5$ by $1QC$, &c. then

$$\left.\begin{array}{l} 1Q - 2 = PC \\ 1C - 3N = PD \\ 1QQ - 4Q + 2 = PE \\ 1QC - 5C + 5N = PF \\ \text{* \&c.} \end{array}\right\} \text{ or in modern notation, } \left\{\begin{array}{l} N^2 - 2 = PC \\ N^3 - 3N = PD \\ N^4 - 4N^2 + 2 = PE \\ N^5 - 5N^3 + 5N = PF, \\ \text{\&c.} \end{array}\right.$$

but, $N^2 - 2$, $N^3 - 3N$, &c. express the sums, of the squares, of the cubes, &c. of the roots of an equation $x^2 - Nx + 1$; for the formula for the sum of the m^{th} powers is

$$N^m - mN^{m-2} + m \cdot \frac{m-3}{2} N^{m-4} - m \cdot \frac{m-4}{2} \cdot \frac{m-5}{3} N^{m-6} + \text{\&c.}$$

Vieta, therefore, is not to be entirely excluded from the honour due to the invention of the preceding theorem.

Vieta calculated by chords; and, his formulæ, which we have just given, are, in fact, the same as the expressions for cos. $2A$, cos. $3A$, &c. given in pages 39 and 40.

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* See Simpson's *Essays*, p. 106.

Vieta also has, p. 297, given another form, exhibiting the relations between the chords of AB, AC, AD, &c. He puts the chord of $AB = 1$ and the relation of the chord of AC to the chord of AB, N, consequently, $N = \frac{\text{chord } AC}{\text{chord } AB} =$

$$\frac{2 \cdot \sin. \frac{AC}{2}}{2 \cdot \sin. \frac{AB}{2}} = \frac{\sin. AB}{\sin. \frac{AB}{2}} = \frac{2 \sin. \frac{AB}{2} \cdot \cos. \frac{AB}{2}}{\sin. \frac{AB}{2}} = 2 \cos. \frac{AB}{2}$$

He then forms this Table:

$1\,Q - 1 = \text{chord } AD$		$N^2 - 1 = \text{chord } AD$
$1\,C - 2\,N = \text{chord } AE$	or in modern notation,	$N^3 - 2\,N = \text{chord } AE$
$1\,QQ - 3\,Q + 1 = \text{chord } AF$		$N^4 - 3\,N^2 + 1 = \text{chord } AF$
&c.		&c.

Now since the chord $AD = 2 \sin. 3\left(\frac{AB}{2}\right)$, chord $AE = 2 \sin. 4\left(\frac{AB}{2}\right)$ &c. and since $2 \sin. \frac{AB}{2}$ is put $= 1$, the preceding formulæ become, if we put $\frac{AB}{2} = A$,

$$(2 \cos. A)^2 - 1 = 2 \sin. 3A,$$
$$(2 \cos. A)^3 - 2 \times 2 \cdot \cos. A = 2 \cdot \sin. 4A,$$
&c.

which are the same, in fact, as $[s^{II}]$ $[s^{IV}]$ given in page 41.

We may also employ the above mode of expressing the cosines of multiple arcs, in deducing de Moivre's formula, which is

* $(\cos. A + \sqrt{-1} \sin. A)^m = \cos. mA + \sqrt{-1} \cdot \sin. mA$, for since

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* Lagrange, p. 116 *Calcul des Fonctions*, says, that this form is as remarkable for its simplicity and elegance, as it is for its generality and utility: and M. Laplace, in the *Leçons des Ecoles Normales*, considers the invention of this formula to be of equal importance with that of the Binomial Theorem.

$$\cos. A = \frac{1}{2}\left(x + \frac{1}{x}\right), \quad \sin. A = \frac{1}{2}\sqrt{\left(-x^2 + 2 - \frac{1}{x^2}\right)},$$

$$= \frac{\sqrt{-1}}{2}\left(x - \frac{1}{x}\right), \text{ or } 2\sqrt{-1}\sin. A = \left(x - \frac{1}{x}\right),$$

and similarly, $2\sqrt{-1}\cdot\sin. mA = x^m - \frac{1}{x^m}$;

hence, $\cos. A + \sqrt{-1}\sin. A = x$, $\cos. A - \sqrt{-1}\cdot\sin. A = \frac{1}{x}$,

and, $\cos. mA + \sqrt{-1}\cdot\sin. mA = x^m$, $\cos. mA - \sqrt{-1}\sin. mA = \frac{1}{x^m}$

consequently,

$$(\cos. A + \sqrt{-1}\sin. A)^m = \cos. mA + \sqrt{-1}\cdot\sin. mA,$$

and $(\cos. A - \sqrt{-1}\cdot\sin. A)^m = \cos. mA - \sqrt{-1}\cdot\sin. mA.$

If we expand these expressions, and then add them, we shall have

$$\cos. mA =$$

$$\cos.^m A - \frac{m\cdot(m-1)}{2}\cos.^{m-2} A \times \sin.^2 A + \frac{m(m-1)(m-2)(m-3)}{2\cdot 3\cdot 4} \times$$

$$\cos.^{m-4} A \times \sin.^4 A - \&c.$$

If we subtract them

$$\sin. mA =$$

$$m\cdot\cos.^{m-1} A \times \sin. A - \frac{m\cdot(m-1)(m-2)}{2\cdot 3}\cos.^{m-3} A \times \sin.^3 A + \&c.$$

From the above mode of representing the cosines of multiple arcs we may also deduce, and concisely, the formulæ of Cotes, page 113, &c. * *Theor. Log. Præf. in Harmonia Mensurarum*, and of De Moivre, *Misc. Analyt.* p. 16, &c. thus,

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* The Theorem of Cotes was not announced to the public by its Author, but by the Editor of his Works, Dr. Robert Smith, who informs us, page 113, Preface, that after various conjectures and trials, he extracted it and its meaning from the deceased Author's loose papers "*Revocavi tandem ab interitu Theorema Pulcherrimum.*" M. Lagrange conjectures, and with probability, that Cotes arrived at his Theorem by the way of Vieta's Theorems. See page 47.

In the expression,

$$2 \cdot \cos. mA = x^m + \frac{1}{x^m}, \text{ make } mA = \theta;$$

then $x^m + \frac{1}{x^m} = 2 \cos. \theta$, or, $x^{2m} - 2 \cdot \cos. \theta \times x^m + 1 = 0$,

and $x + \frac{1}{x} = 2 \cdot \cos. \frac{\theta}{m}$, or, $x^2 - 2 \cdot \cos. \frac{\theta}{m} x + 1 = 0$.

Now from $x + \frac{1}{x} = 2 \cdot \cos. \frac{\theta}{m}$, $x^m + \frac{1}{x^m} = 2 \cos. \theta$ was deduced; therefore, if x were deduced from the first expression in terms of $\cos. \frac{\theta}{m}$, or, in other words, if a were the root of the equation, $x^2 - 2 \cos. \frac{\theta}{m} x + 1 = 0$, that same value of x, or root a, substituted in the second expression, $x^m + \frac{1}{x^m} = 2 \cos. \theta$, would make it a true equation, or a would be a root of the equation

$$x^{2m} - 2 \cos. \theta . x^m + 1 = 0.$$

Hence, by the doctrine of equations, $x - a$, is a divisor, both of $x^2 - 2 \cdot \cos. \frac{\theta}{m} x + 1$, and of $x^{2m} - 2 \cos. \theta . x^m + 1$; and similarly, since $\frac{1}{a}$ is the other root of $x^2 - 2 \cos. \frac{\theta}{m} x + 1$, provided a be one root, $x - \frac{1}{a}$ is a divisor both of $x^2 - 2 \cos. \frac{\theta}{m} x + 1$, and of $x^{2m} - 2 \cos. \theta . x^m + 1$; and consequently, $(x - a)\left(x - \frac{1}{a}\right)$ or, $x^2 - 2 \cdot \cos. \frac{\theta}{m} x + 1$ is a divisor of

$$x^{2m} - 2 \cos. \theta . x^m + 1.$$

Now, by Table, p. 9, and by the preceding reasoning, it appears that the arcs

$$2\pi - \theta, \quad 4\pi - \theta, \text{ &c.} \quad (2n + 2)\pi - \theta,$$
$$2\pi + \theta, \quad 4\pi + \theta, \text{ &c.} \quad 2n\pi + \theta,$$

have the same cosine as the arc θ has: but if, instead of the equation

$x + \frac{1}{x} = 2\cos.\frac{\theta}{m}$, we assume

$$x+\frac{1}{x}=2.\cos.\left(\frac{2\pi+\theta}{m}\right) \text{ or } = 2.\cos.\left(\frac{4\pi+\theta}{m}\right) \text{ or } = 2\cos.\left(\frac{2n\pi+\theta}{m}\right)$$

$$\text{or} = 2.\cos.\left(\frac{2\pi-\theta}{m}\right) \text{ or } = 2.\cos.\left(\frac{4\pi-\theta}{m}\right) \text{or} = 2\cos.\left(\frac{(2n+2)\pi-\theta}{m}\right)$$

the resulting expressions will be respectively

$$x^m + \frac{1}{x^m} = 2\cos.(2\pi+\theta) \text{ or } = 2.\cos.(4\pi+\theta) \text{ or } = 2\cos.(2n\pi+\theta);$$

$$\text{or } = 2.\cos.(2\pi-\theta), \text{ or } = 2\cos.(4\pi-\theta), \text{ or } = 2.\cos.(2n+2)\pi-\theta,$$

which expressions, by what has just appeared, (p. 50, l. 19, &c.) are all of equal value.

Hence, of the same expression

$$x^{2m} - 2.\cos.\theta.x^m + 1,$$

$$x^2 - 2\cos.\frac{\theta}{m}x + 1,\ x^2 - 2\cos.\frac{2\pi+\theta}{m}x + 1,$$

$$x^2 - 2.\cos.\left(\frac{2\pi-\theta}{m}\right)x + 1,\ x^2 - 2.\cos.\frac{4\pi+\theta}{m}x + 1, \text{ &c.}$$

are divisors; in other words, $x^{2m} - 2\cos.\theta.x^m + 1$ may be represented by a product, of which these latter quantities are the factors; accordingly

$$x^{2m} - 2\cos.\theta.x^m + 1 =$$

$$\left(x^2 - 2.\cos.\frac{\theta}{m}x + 1\right) \times \left(x^2 - 2.\cos.\frac{2\pi+\theta}{m}x + 1\right) \times$$

$$\left(x^2 - 2.\cos.\frac{2\pi-\theta}{m}x + 1\right) \times \text{ &c.}$$

If we make $\theta = 0$, we have $\cos.\theta = 1$, and $\cos.\frac{2\pi+\theta}{m} = \cos.\frac{2\pi-\theta}{m}$

and $x^{2m} - 2x^m + 1$, or $(x^m - 1)^2 =$

$$(x^2 - 2x + 1).\left(x^2 - 2\cos.\frac{2\pi}{m}x + 1\right)^2\left(x^2 - 2.\cos.\frac{4\pi}{m}x + 1\right)^2 \times$$

&c.

and accordingly,

$$x^m - 1 =$$

$$(x-1).\left(x^2-2.\cos.\frac{2\pi}{m}x+1\right).\left(x^2-2\cos.\frac{4\pi}{m}x+1\right),\ \&c.$$

which is the analytical expression of Cotes's Theorem. See *Harmonia Mensurarum*, p. 114, &c. and De Moivre's *Miscellanea Analytica*, p. 17, &c.

We will now proceed to investigate expressions for the powers of the cosine and sine of an arc, in terms of the cosines and sines of multiple arcs, which expressions are highly useful in all Mathematical Investigations connected with Physical Astronomy.

PROBLEM 7. It is required to express the powers of the cosine and sine of an arc, in terms involving the cosines and sines of the multiple arc.

$$2.\cos.A=x+\frac{1}{x};$$

$$\therefore 2^n\cos.^n A=\left(x+\frac{1}{x}\right)^n=$$

$$*\ x^n+n\,x^{n-2}+n.\left(\frac{n-1}{2}\right)x^{n-4}+\&c.\ \frac{n}{x^{n-2}}+\frac{1}{x^n}=$$

(collecting into pairs the terms equidistant from each extremity of the series)

$$\left(x^n+\frac{1}{x^n}\right)+n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n.\frac{n-1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\&c.$$

$\therefore$ by page 42, $2^{n-1}\cos.^n A$

$$=\cos.nA+n.\cos.(n-2)A+n.\frac{n-1}{2}\cos.(n-4)A+\&c.$$

The number of terms is $n+1$, therefore if n be even, the last term is

$$\frac{n.(n-1)(n-2).....\left(n-\frac{n}{2}+1\right)}{1.2.3.....\frac{n}{2}}\times\frac{1}{2}\cos.(n-n)A.$$

* For the expansion of the Binomial, see Wood's *Algebra*, page 109, first edition; or Vince's *Fluxions*, p. 45; or Woodhouse's *Principles of Analytical Calculation*, page 24, &c.

Now,

$$\frac{n\,.\,(n-1)\,(n-2)\,.\,.\,.\,.\,.\left(n-\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\frac{n}{2}} = \frac{2n\,(2n-2)\,.\,.\,.\,2\left(n-\frac{n}{2}+1\right)}{2\,.\,4\,.\,.\,6\,.\,.\,.\,.\,.\,n}$$

$$= \frac{2n\,.\,(2n-2)\,.\,\&c}{2\,.\,4\,.\,.\,.\,.\,.\,.\,.\,.\,.\,n} \times \frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,.\,.\,.\,(n-1)}{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,.\,.\,.\,(n-1)} =$$

$$2\times2\times2,\ \&c.\left(\text{to } \frac{n}{2} \text{ terms}\right) \times \frac{n\,.\,(n-1)\,(n-2)\,.\,.\left(\frac{n}{2}+1\right)}{1\,.\,2\,.\,3,\ \&c.\,.\,.\,(n-1)\,n} \times 1\,.\,3\,.\,5$$

$$\ldots(n-1) = 2^{\frac{n}{2}} \times \frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.(n-1)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,\frac{n}{2}};$$

consequently, since $\cos.\,(n-n)\,A = 1$, the last term $=$

$$2^{\frac{n}{2}-1} \times \frac{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,.\,.\,(n-1)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,.\,.\,.\,\frac{n}{2}}.$$

Hence, as instances of the general form,

$n=2$, $2\,.\,\cos.^2 A = \cos.\,2A + 1$ ..... $[e^{II}]$
$n=3$, $2^2\,.\,\cos.^3 A = \cos.\,3A + 3\,.\,\cos.\,A$ ..... $[e^{III}]$
$n=4$, $2^3\,.\,\cos.^4 A = \cos.\,4A + 4\,.\,\cos.\,2A + 3$ ..... $[e^{IV}]$
$n=5$, $2^4\,.\,\cos.^5 A = \cos.\,5A + 5\,\cos.\,3A + 10\,\cos.\,A$ ..... $[e^{V}]$
$n=6$, $2^5\,.\,\cos.^6 A = \cos.\,6A + 6\,\cos.\,4A + 15\,.\,\cos.\,2A + 10$ $[e^{VI}]$
&c. &c.

In order to obtain a general form for $\sin.^n A$, in the form for $\cos.^n A$ substitute instead of A, $\frac{\pi}{2} - A$, then

$$2^{n-1}\,.\,\cos.^n\left(\frac{\pi}{2}-A\right) = \cos.\left(n\frac{\pi}{2} - nA\right) + n\,.\,\cos.\,(n-2)\left(\frac{\pi}{2}-A\right)$$

$$+ \frac{n\,.\,n-1}{2}\,.\cos.\,(n-4)\left(\frac{\pi}{2}-A\right) + \&c.$$

Now, let n be even, and of the form $2^{2m}\,.\,p$, p being an odd number, or let n be, as it is called, *pariter par*;

then $\frac{n\pi}{2} = 2^{2m-1}p \cdot \pi = 2^{2m-2}p \cdot 2\pi$, but $2^{2m-2}.p$ is a whole number, and therefore cos. $(2^{m-2}p \cdot 2\pi)$ is $= 1$.

Again, $(n-2)\frac{\pi}{2} = (2^{2m}p - 2)\frac{\pi}{2} = (2^{2m-1}p - 1)\pi$

but $(2^{2m-1}p - 1)$ is an odd number, and $\therefore$ cos. $(2^{2m-1}p - 1)\pi = -1$; hence, since cos. $\left(\frac{\pi}{2} - A\right)$ = sin. A, and cos. $\left(\frac{n\pi}{2} - nA\right)$ = cos. $\frac{n\pi}{2}$ cos. nA = cos. nA, [since sin. $\frac{n\pi}{2} = 0$], &c.

$$2^{n-1} \cdot \sin.^n A =$$

$$\cos. nA - n \cdot \cos.(n-2)A + \frac{n \cdot n-1}{2} \cdot \cos.(n-4)A - \&c.$$

If n be even, but of the form $2p$, or *impariter par* $\frac{n\pi}{2} = 2p\frac{\pi}{2} = p\pi$ and cos. $p\pi = -1$, and consequently,

since cos. $(p\pi - nA)$ = cos. $p\pi$. cos. $nA = -$ cos. nA

$$2^{n-1} \cdot \sin.^n A = -\cos. nA + n \cdot \cos.(n-2)A - \&c.$$

and, in both these cases, the last term is, as before,

$$\frac{1 \cdot 3 \cdot 5 \ldots\ldots (n-1)}{1 \cdot 2 \cdot 3 \ldots\ldots \frac{n}{2}} \cdot 2^{\frac{n}{2}} - 1$$

when n is odd, cos. $\frac{n\pi}{2} = 0$, and cos. $\left(\frac{n\pi}{2} - nA\right)$ = sin. $\frac{n\pi}{2}$ sin. nA $= \pm$ sin. nA, where the upper sign is to be taken, if n be 1, 5, 9, &c. Hence,

$$\pm 2^{n-1} \cdot \sin.^n A = \sin. nA - n \cdot \sin.(n-2)A + \frac{n \cdot (n-1)}{1 \cdot 2} \sin.(n-4)A - \&c.$$

Hence, as instances of the general form,

$n = 2$, $2 \cdot \sin.^2 A = -\cos. 2A + 1$ ..... $[f^{II}]$
$n = 3$, $2^2 \cdot \sin.^3 A = -\sin. 3A + 3 \cdot \sin. A$ ..... $[f^{III}]$
$n = 4$, $2^3 \cdot \sin.^4 A = \cos. 4A - 4 \cdot \cos. 2A + 3$ ..... $[f^{IV}]$
$n = 5$, $2^4 \cdot \sin.^5 A = \sin. 5A - 5 \cdot \sin. 3A + 10 \sin. A$ ..... $[f^{V}]$
$n = 6$, $2^5 \sin.^6 A = -\cos. 6A + 6 \cos. 4A - 15 \cos. 2A + 10$... $[f^{VI}]$

PROBLEM 8. It is required to express the tangent of twice, thrice, &c. an arc in terms of the tangent of the simple arc.

By Prob. 4, page 29,

$$\text{tan. } (A + B) = \frac{\text{tan. } A + \text{tan. } B}{1 - \text{tan. } A . \text{tan. } B};$$

consequently,

$$\text{tan. } (A+A), \text{ or, tan. } 2A = \frac{\text{tan. } A + \text{tan. } A}{1 - \text{tan. } A \text{ tan. } A} = \frac{2 \text{ tan. } A}{1 - \text{tan.}^2 A}.$$

$$\text{Again,} \quad \text{tan. } (2A + A) = \frac{\text{tan. } 2A + \text{tan. } A}{1 - \text{tan. } 2A . \text{tan. } A},$$

and the numerator,

$$\text{tan. } 2A + \text{tan. } A = \frac{2 \text{ tan. } A}{1 - \text{tan.}^2 A} + \text{tan. } A = \frac{3 \text{ tan. } A - \text{tan.}^3 A}{1 - \text{tan.}^2 A}$$

the denominator, which is

$$1 - \text{tan. } A . \text{tan. } 2A = 1 - \frac{2 . \text{tan.}^2 A}{1 - \text{tan.}^2 A} = \frac{1 - 3 . \text{tan.}^2 A}{1 - \text{tan.}^2 A}.$$

Hence

$$\text{tan. } 3A \left\{ = \text{tan. } (2A + A) \right\} = \frac{3 . \text{tan. } A - \text{tan.}^3 A}{1 - 3 \text{ tan.}^2 A},$$

and, by a similar method, since $4A = 3A + A$, and $5A = 4A + A$,

$$\text{tan. } 4A = \frac{4 \text{ tan. } A - 4 . \text{tan.}^3 A}{1 - 6 . \text{tan.}^2 A + \text{tan.}^4 A},$$

$$\text{tan. } 5A = \frac{5 \text{ tan. } A - 10 . \text{tan.}^3 A + \text{tan.}^5 A}{1 - 10 \text{ tan.}^2 A + 5 \text{ tan.}^4 A}.$$

These expressions for the tangents of multiple arcs may also be derived from those given in p. 31. For instance, since A, B, C, D are equal, t, t', t'', t''' are, and $S_4 (t) = 4t$

$$S_4 (tt't'') = \frac{4 \times 3 \times 2}{1 \times 2 \times 3} t^3 = 4t^3,$$

$$S_4 (tt') = \frac{4 \times 3}{1 . 2} t^2 = 6t^2$$

$$S_4 (tt't''t''') = t^4$$

consequently,

$$\text{tan. } (A + A + A + A) = \text{tan. } 4A = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

as before, p. 55, l. 16.

In like manner, we may deduce tan. $5A$ from the expression for tan. $(A + B + C + D + E)$ (see p. 31, l. 20.)

For,

$$S_5\,(t) = 5t,$$

$$S_5\,(tt't'') = \frac{5 \times 4 \times 3}{1 \times 2 \times 3}\,t^3 = 10t^3$$

$$S_5\,(tt' \ldots t^{IV}) = t^5$$

$$S_5\,(tt') = \frac{5 \times 4}{1 \times 2}\,t^2 = 10t^2$$

$$S_5\,(tt't''t''') = \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4}\,t^4 = 5t^4;$$

$$\therefore \text{tan. } 5A = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

as before.

It is not, perhaps, necessary to multiply farther trigonometrical formulæ; such as are chiefly useful, and usually occur in investigation, have been given; and the Student, who thoroughly apprehends the principle and mode of their deduction, will be able, by his own dexterity, to deduce others.

A sufficient number of formulæ having been given, it may now be thought proper to proceed to their application; and, the first object of their application seems naturally to be that for which the science of Trigonometry was originally invented; the Solution of Rectilinear Triangles. Now, this solution consists of two parts; first, it is necessary to express the relations of the sides and angles of triangles by Trigonometrical symbols; and, secondly, to afford the means of arithmetically computing, in specific instances, the values of such symbols. For instance, if two sides, a, b, and an angle A of a rectilinear triangle should be given, the value of the angle B (see p. 22.) would be truly expressed by

$$\text{sin. } B = \text{sin. } A \times \frac{b}{a}.$$

But, this is an algebraical value; in order to obtain a practical result, we must be able, when b and a are expounded by numbers, and A by degrees, minutes, &c. to express B in degrees, minutes, &c. we must, therefore, possess the means of assigning sin. A from a given value of A, and also of assigning B from a resulting value of sin. B. These means, in practice, are afforded by Trigonometrical Tables, and their formation, or, what technically is called the construction of the *Trigonometrical Canon*, is an easy consequence from the preceding results.

We will, in the next Chapter, proceed to the construction of this Canon, which may be viewed either as a distinct application of the preceding formulæ, or as a preparatory step to their application in the solution of rectilinear triangles.

CHAP. IV.

On the Construction of the Trigonometrical Canon.—Methods of computing the Sine of 1′.— The Sines and Cosines of successive Arcs. — Formulæ of Verification, or Methods of examining the Accuracy of Computed Tables.

PROBLEM 9. IT is required to find a numerical value of the sine of 1 minute; the circle being divided into 360 × 60 or 21600 minutes, and its radius being 1.

By Cor. 3, Prob. 5, cos. $A = \sqrt{\{\frac{1}{2}(1 + \cos. 2A)\}}$ [a], and by Cor. 2, Prob. 5, if $2A = 60^\circ$, cos. $2A = \frac{1}{2} = .5$,

consequently, cos. $30^\circ = \sqrt{\{\frac{1}{2}(1 + \frac{1}{2})\}} = .8660254$;

substitute this value into the form [a], and we shall have cos. 15°: and, by a repetition of the operation, successively, the cosines of $\frac{60^\circ}{2^3}$, $\frac{60^\circ}{2^4}$, $\frac{60^\circ}{2^5}$, $\frac{60^\circ}{2^{12}}$: so that, if c, c^I, c^{II}, &c. stand for the successive cosines, the operation may be thus exhibited:

cos. $\frac{60^\circ}{2}$, or, cos. $30^\circ = \sqrt{\left\{\frac{1}{2}\left(1 + \frac{1}{2}\right)\right\}} = c^I = .8660254$, &c.

cos. $\frac{60^\circ}{2^2}$, or, cos. $15^\circ = \sqrt{\left(\frac{1}{2}(1 + c^I)\right)} = c^{II} = .9659258$, &c.

cos. $\frac{60^\circ}{2^3}$, or, cos. $7^\circ\ 30' = \sqrt{\left(\frac{1}{2}(1 + c^{II})\right)} = c^{III} = .9914449$, &c.

cos. $\frac{60^\circ}{2^4}$, or, cos. $3^\circ\ 45' = \sqrt{\left(\frac{1}{2}(1 + c^{III})\right)} = c^{IV} = .9978589$, &c.

&c. &c.

* $\cos.\frac{60^\circ}{2^{12}}$, or, $\cos.\ 52''\ 44'''\ .3^{IV}\ .45^{V} = \sqrt{\left(\frac{1}{2}(1+c^{XI})\right)} =$ .99999996732.

From this value of the cosine, the sine $=$ .000255663462.

In order to find the sin. 1′, we must compute on this principle, that the sines of very small arcs are to one another as the arcs themselves †; and then, since the arcs $52''\ .\ 44'''\ .\ 3^{IV}\ .\ 45^{V}$, and 1′, are to one another as $\frac{60}{2^{12}} : \frac{60}{60\times 60}$;

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* The value of $\frac{60^\circ}{2^{12}}$ may be shortly obtained by the following mode of decomposing numbers:

$$\frac{60^\circ}{2^{12}} = \frac{60\ .\ 60\ .\ 60}{2^{12}} \text{ (the units being seconds)} = \frac{(64-4)\ (64-4)\ (64-4)}{2^{12}}$$

$$= \frac{(2^6-2^2)^3}{2^{12}} = \frac{(2^4-1)^3}{2^6} = \left(\frac{2^4-1}{2^2}\right)^3 = \left(2^2-\frac{1}{2^2}\right)^3 = 2^6 - 3\ .\ 2^2 + \frac{3}{2^2} - \frac{1}{2^6},$$

$$\text{but } 2^6 + \frac{3}{2^2} = 64''\ .\ 45''',$$

$$3\ .\ 2^2 + \frac{1}{2^6} = 12''\ .\ 0\ .\ 56^{IV}\ .\ 15^{V},$$

$$\text{subtracting, } \frac{60^\circ}{2^{12}} = 52''\ .\ 44'''\ .\ 3^{IV}\ .\ 45^{V}.$$

† If from the two extremities of an arc there be drawn two tangents these will be equal. By the principle assumed by Archimedes, the arc is $<$ sum of tangents $>$ chord joining the two ends of the arc; consequently $\frac{1}{2}$ arc $<$ tangent $>$ $\frac{1}{2}$ chord or $>$ sine of $\frac{1}{2}$ arc: and therefore $\tan.\frac{A}{2} - \sin.\frac{A}{2} > \frac{A}{2} - \sin.\frac{A}{2}$: hence, if, in the instance given in the text, we find the difference between the tangent and sine of 1′ (computing tan. 1′ from $\frac{\sin.\ 1'}{\cos.\ 1'}$), it will be found to be (.000290888216 $-$.000290888204 $=$ .000000000012: consequently, the arc of 1′ differs from its sine by a quantity less than .000000000012: so that, it is plain, the principle of very small arcs varying as their sines, is very little remote from the truth; or rather, if assumed, will entail on the computation a very small numerical error.

or, as 3600 : 4096,

we have, sin. $1' = .000255663462 \times \frac{4096}{3600} = .000290888204$.

This method of computing the sine of 1′ is very operose, but of no great difficulty; for, it requires only the knowledge of the simplest arithmetical operations. But, even if we avail ourselves of the formulæ and inventions of the analytic art, the computation of sin. 1′ cannot, in any case, be very expeditiously effected; more expeditiously, however, than by the preceding method.

If we employ the expression for the sin. A which was given, in page 38, we may successively deduce the sines of 30°, 15°, 7° 30′, &c. by operations analogous to those already given; thus, since sin. $A = \frac{1}{2}\sqrt{(1 + \text{sin. } 2A)} - \frac{1}{2}\sqrt{(1 - \text{sin. } 2A)}$, and sin. $30° = \frac{1}{2}$,

$A = 30°$	sin. $30° = \frac{1}{2} \ldots\ldots = .5$
$A = 15°$	sin. $15° = \frac{1}{2}\sqrt{(1 + \frac{1}{2})} - \frac{1}{2}\sqrt{(1 - \frac{1}{2})} = s' = .258819$
$A = 7° 30'$	sin. $7° 30' = \frac{1}{2}\sqrt{(1 + s')} - \frac{1}{2}\sqrt{(1 - s')} = s'' = .1305262$
$A = 3° 45'$	sin. $3° 45' = \frac{1}{2}\sqrt{(1 + s'')} - \frac{1}{2}\sqrt{(1 - s'')} = s''' = .0654031$

and so on we may compute till we obtain the sin. $\frac{30°}{2^{11}}$.

The preceding computation was made to begin from $A = 30°$, because the sin. 30° is known; but we might have begun from any other arc, the sine of which is known: thus, if we take $2A = 18°$, the sine of $18° = \frac{1}{2}$ chord 36°, but the chord 36° = the side of a regular decagon inscribed in a circle, $= BD$ the base of the isoceles triangle described in the 10th Proposition 4th Book of Euclid, * $= \frac{\sqrt{5} - 1}{2}$: hence, sin. $18° = \frac{\sqrt{5} - 1}{4}$, and substituting this value for sin. $2A$ in the above form, we have

sin. 9°	$= \frac{1}{4}\sqrt{(3 + \sqrt{5})} - \frac{1}{4}\sqrt{(5 - \sqrt{5})} = s = .156434$
sin. 4° 30′	$= \frac{1}{2}\sqrt{(1 + s)} - \frac{1}{2}\sqrt{(1 - s)} = s' = .0784591$
sin. 2° 15′	$= \frac{1}{2}\sqrt{(1 + s')} - \frac{1}{2}\sqrt{(1 - s')} = s'' = .0392598$
&c.	= &c.

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* $BAD + 2ABD = 180°$, but $ABD = 2BAD$; $\therefore 5BAD = 180°$; $\therefore 10BAD = 360°$ and $BAD = 36°$: again, by the Prop. $AB \cdot BC = AC^2 = BD^2$, consequently, if $BD = x$, $AB = 1$, $1 \cdot (1 - x) = x^2$, and $x = \frac{\sqrt{5} - 1}{2}$.

Having thus computed the sine of a small arc that is nearly 1′, the method of determining the sin. 1′ is, in principle, precisely the same as in Prob. 9, p. 58. There is however, a third method, considerably different in its principle, and which finds the sine of 1′ by the quinquesection and trisection of an arc. It may be thus explained,

By the form [s^V], page 41,

$$\sin. 5A = 5 \sin. A - 20 \sin.^3 A + 16 \sin.^5 A.$$

Let $5A = 30°$ then $\sin. 5A = \frac{1}{2}$ and $A = 6°$; let $2 \sin. A = x$, then

$$1 = 5x - 5x^3 + x^5.$$

By approximation, find the value of x: thus, suppose a to be a near value, and $a + v$ to be the true, then

$$1 = 5(a+v) - 5(a^3 + 3a^2v) + a^5 + 5a^4v,$$

neglecting the terms that involve v^2, v^3, &c. consequently,

$$v = \frac{1 - 5a + 5a^3 - a^5}{5(1 - 3a^2 + a^4)}.$$

Now, since $\sin. 5A = \frac{1}{2}$, assume, as a first approximate value of x, $a = \frac{2}{10} = .2$: substitute in the expression for v this value, and find the resulting value of v; it will appear to be $= .009$, the corrected value of a then, or $a + v$, is $.209$; with this, find a new value of v, and another corrected value of a, and repeat the operation till x is found exact to a certain number of decimals, seven for instance; in which case

$x = .2090569$, and consequently $\sin. A = .1045285$.

Having thus obtained the sine of 6°, in the form [s'''] page 41, that is, in $\sin. 3A = 3 \sin. A - 4 \sin.^3 A$, put $3A = 6°$, and $2 \sin. A = x$, then the equation becomes $.2090569 = 3x - x^3$. Find, as before, by the method and formula of approximation, a value of x, which, to seven places of figures, will be $.0697989$, consequently x or $2 \sin. 2° = .0697989$, and $\sin. 2° = .0348995$.

In order to find sin. 1°, take the form p. 36, $\sin. 2A = 2 \sin. A \cos. A$, then $\sin.^2 2A = 4 . \sin.^2 A - 4 . \sin.^4 A$: substitute for

$\sin.^2 2A$, or $\sin.^2 2°$, its value, and by the solution of a quadratic find $\sin.^2 A$, and then, $\sin. A$, or $\sin. 1°$, the value of which to seven places of figures is .0174524. Repeat the operation, and we have $\sin. \frac{1°}{2}$, or $\sin. 30'$, the value of which is .0087265.

By this method then we have descended from the $\sin 30°$ to the $\sin. 1°$ and $\sin. 30'$; and consequently, by like operations, we can descend from $\sin. 30'$ to $\sin. 1'$ and $\sin. 30''$: and by this method, which is however extremely operose, we are able to find the $\sin. 1'$ without a proportion, and accordingly to avoid the use of a principle, which some may think doubtful; namely, that the sines of small arcs are to one another as the arcs themselves.

The above method is, in fact, the same as that which is given with all its detail, at page 451, &c. in the sixth Volume of the *Scriptores Logarithmici*, edited by Baron Maseres. It is plain, however, that there is no necessity for beginning the computation from an arc of 30°; we may make it begin from any arc, the sine of which is known: for instance, by the form, page 65,

$$\sin. 9° = \tfrac{1}{4}\sqrt{(3 + \sqrt{5})} - \tfrac{1}{4}\sqrt{(5 - \sqrt{5})} = .156434;$$

since, therefore, $\sin. 9° = \sin. (3 . 3°) = 3 \sin. 3° - 4 \sin.^3 3°$, solve, as before, by approximation, the equation

$$.156434 = 3x - 4x^3,$$

and the result gives x the $\sin. 3°$. Again, solve a similar equation by the same mode and formula, and the result gives $\sin. 1°$. And many like methods will suggest themselves to the mind of the intelligent Student.

We shall now proceed to the second part of the construction of Trigonometrical Tables, the object of which will be understood from the succeeding Problem.

PROBLEM 10. It is required from the $\sin. 30''$ and $\sin. 1'$, to compute the sines of 2, 3, 4, &c. minutes, and also the sines of 1, 2, 3, &c. degrees.

By the form [*a*], page 26,

$$\sin. (A + B) = 2 \sin. A . \cos. B - \sin. (A - B)$$

$$= 2 \sin. A \left(1 - 2 \sin.^2 \frac{B}{2}\right) - \sin. (A - B)$$

$$= \sin. A + \left\{\sin. A - \sin. (A - B)\right\} - 4 \sin. A \sin.^2 \frac{B}{2}.$$

If $B = 1°$, then

$\sin. (A + 1°) = \sin. A + \sin. A - \sin. (A - 1°) - 4 \sin. A \sin.^2 30'$ which is Delambre's formula.

Let $B = 1'$ and let A successively equal $1'$, $2'$, $3'$, &c. then

$\sin. 2' = \sin. 1' + (\sin. 1' - \sin. 0) - 4 \sin. 1'. (\sin. 30'')^2$
$\sin. 3' = \sin. 2' + (\sin. 2' - \sin. 1') - 4 \sin. 2'. (\sin. 30'')^2$
$\sin. 4' = \sin. 3' + (\sin. 3' - \sin. 2') - 4 \sin. 3'. (\sin. 30'')^2$
$\sin. 5' = \sin. 4' + (\sin. 4' - \sin. 3') - 4 \sin. 4'. (\sin. 30'')^2$,
&c. = &c.

and thus may the sines of all succeeding arcs be computed, by a process not very tedious, since the only part of it at all long is the multiplication of sin. $2'$, sin. $3'$, &c. by the constant factor $(2 \sin. 30'')^2$, which is the square of the chord of $1'$.

In the above form, substitute instead of B, $1°$, and instead of A successively 1, 2, 3, &c. degrees; then

$\sin. 2° = \sin. 1° + (\sin. 1° - \sin. 0) - 4 \sin. 1° (\sin. 30')^2$
$\sin. 3° = \sin. 2° + (\sin. 2° - \sin. 1°) - 4 \sin. 2°. (\sin. 30'')^2$,
&c.

and so on for the sines of all succeeding arcs.

In order to compute the sines of arcs composed of degrees and minutes; arcs, for instance, such as $3° 2'$, $3° 3'$, substitute for B, $1'$, and for A successively $3° 1'$, $3° 2'$, $3° 3'$, &c. then

$\sin. 3° 2' = \sin. 3° 1' + (\sin. 3° 1' - \sin. 3°) - 4 \sin. 3° 1' (\sin. 30'')^2$
$\sin. 3° 3' = \sin. 3° 2' + (\sin. 3° 2' - \sin. 3° 1') - 4 \sin. 3° 2' (\sin. 30'')^2$
&c.

or, if we wish to compute for every ten minutes, put $B = 10'$, and for A write successively $A + 10'$, $A + 20'$, $A + 30'$, &c. thus, if $A = 7°$

$\sin. 7° 20' = \sin. 7° 10' + (\sin. 7° 10' - \sin. 7°) - 4 \sin. 7° 10'. (\sin. 5')^2$
$\sin. 7° 30' = \sin. 7° 20' + (\sin. 7° 20' - \sin. 7° 10') - 4 \sin. 7° 20'. (\sin. 5')^2$
&c.

By the preceding methods we are enabled regularly to compute the sines of all arcs from 1′ or 1″ up to 90°; but, when the arcs exceed 60°, the application of the Trigonometrical formula, page 26, renders the arithmetical computation more simple and concise: thus, since

$$\sin.(B+A) = \sin.(B-A) + 2 \cdot \cos. B. \sin. A.$$

Let $B = 60°$, then $\cos. B = \cos. 60° = \frac{1}{2}$, consequently

$$\sin.(60° + A) = \sin. A + \sin.(60° - A).$$

(see form 15 of Table, p. 33.)

Hence, instead of the preceding, we may use this latter method, and compute the sines of all arcs exceeding 60°, by the simple addition of the sines of arcs previously computed: for instance,

$$\sin. 63° 5′ = \sin. 3° 5′ + \sin. 56° 55′$$

and, since sin. 3° 5′ = .0537883, and sin. 56° 55′ = .8378775

the sin. 63° 5′ = .891665.

The sines of all arcs from 0 to 90° being computed, the cosines of all the arcs of the quadrant are known; for, $\cos. A = \sin.(90° - A)$; for instance, cos. 63° 15′ 7″ = sin. 26° 44′ 53″ cos. 13° 47′ = sin. 76° 13′, &c.

The sines and cosines being computed, the tangents may be computed from this expression, $\tan. A = \frac{\sin. A}{\cos. A}$, and the cotangents from $\text{cotan. } A = \frac{\cos. A}{\sin. A}$.

When the tangents of arcs up to 45° have been computed, the Trigonometrical formulæ previously given may be conveniently used in computing the tangents of arcs that lie between 45° and 90°: thus, by Prob. 4, page 29,

$$\tan.(A+B) = \frac{\tan. A + \tan. B}{1 - \tan. A. \tan. B} = [\text{if } B = 45°] \frac{1 + \tan. A}{1 - \tan. A};$$

hence, as instances, putting $A = 1°, 2°, 3°$, &c.

$$\tan.(45° + 1°), \text{ or, } \tan. 46° = \frac{1 + \tan. 1°}{1 - \tan. 1°},$$

$$\text{tan. } (45^\circ + 2^\circ), \text{ or, tan. } 47^\circ = \frac{1 + \text{tan. } 2^\circ}{1 - \text{tan. } 2^\circ},$$

&c. &c.

or, we may avoid the fractional form thus,

$$\text{tan. } (45^\circ + A) = \frac{1 + \text{tan. } A}{1 - \text{tan. } A} \text{ and tan. } (45^\circ - A) = \frac{1 - \text{tan. } A}{1 + \text{tan. } A};$$

$$\therefore \text{ tan. } (45^\circ + A) - \text{ tan. } (45^\circ - A) = \frac{(1 + \text{tan. } A)^2 - (1 - \text{tan. } A)^2}{1 - \text{tan.}^2 A}$$

$$= \frac{4 \text{ . tan. } A}{1 - \text{tan.}^2 A};$$

but, by the form of page 55, $\text{tan. } 2A = \frac{2 \text{ tan. } A}{1 - \text{tan.}^2 A}$.

Hence, tan. $(45^\circ + A) = 2$ tan. $2A +$ tan. $(45^\circ - A)$, consequently,

$$\text{tan. } (45^\circ + 1^\circ), \text{ or, tan. } 46^\circ = 2 \text{ tan. } 2^\circ + \text{tan. } 44^\circ,$$
$$\text{tan. } (45^\circ + 2^\circ), \text{ or, tan. } 47^\circ = 2 \text{ tan. } 4^\circ + \text{tan. } 43^\circ,$$
&c.

By these formulæ and methods may the sines, tangents, &c. of arcs be computed. If we attend, however, to the history of the construction of Trigonometrical Tables, we shall find that all Tables have not been computed exactly by the same formulæ and methods: modern Tables, from the improved state of analytic science, have been computed by the most certain and expeditious methods. In the immense *Tables du Cadastre*, formed at the expense of the French Government, the sines of arcs are computed regularly by successive addition, according to the formulæ given in page 63; but, in such a construction, an error committed in the sine of an inferior arc would, it is plain, entail errors on the sines of all succeeding arcs. Hence is created the necessity of some check on the computist, and of some independent mode of examining the accuracy of the computation. For this purpose, formulæ, such as those given in pages 38, 60, derived immediately from established properties, are employed; if the numerical results from these formulæ agree with the results obtained by the regular process of computation, then, it is almost a certain conclusion that the latter process has been rightly conducted.

As there is, in these formulæ, called *Formulæ of Verification*, besides their practical utility, something curious, several are subjoined and proved.

[1] $\sin. 30° = \frac{1}{2}$; $\sin. 45° = \frac{1}{\sqrt{2}}$; $\sin. 60° = \frac{\sqrt{3}}{2}$.

[2] $\sin. 18° = \frac{\sqrt{5}-1}{4}$, by the Note to page 60, or, it may be independently proved thus:

$\sin. 36° = \cos. 54°$, or, $\sin. (2.\ 18°) = \cos. (3.\ 18°)$. Let $x = \cos. 18°$; then by form [c'''], page 39, $\cos. (3.\ 18°) = 4x^3 - 3x$, and by form, p. 36, $\sin. (2.\ 18°) = 2x\sqrt{(1-x^2)} \therefore 4x^3 - 3x$, or, $2\sqrt{(1-x^2)} = 4x^2 - 3$.

This equation, cleared of radicals and reduced, is

$$16x^4 - 20x^2 + 5 = 0;$$

whence $x^2 = \frac{5+\sqrt{5}}{8}$ and $\therefore 1 - x^2 = \frac{3-\sqrt{5}}{8}$ and $\sqrt{1-x^2}$ or $\sin. 18° = \frac{\sqrt{5}-1}{4}$ which is also $\cos. 72°$.

[3] $\sin. 9°$, or, $\cos. 81° = \frac{1}{4}\sqrt{(\sqrt{5}+3)} - \frac{1}{4}\sqrt{(5-\sqrt{5})}$, by form, p. 38, substituting for $\sin. 18°$ its value,

$\cos. 9°$, or, $\sin. 81° = \frac{1}{4}\sqrt{(3+\sqrt{5})} + \frac{1}{4}\sqrt{(5-\sqrt{5})}$ by the same form.

[4] $\sin. 27°$, or, $\cos. 63° = \frac{1}{4}\sqrt{(5+\sqrt{5})} - \frac{1}{4}\sqrt{(3-\sqrt{5})}$
$\cos. 27°$, or, $\sin. 63° = \frac{1}{4}\sqrt{(5+\sqrt{5})} + \frac{1}{4}\sqrt{(3-\sqrt{5})}$

for in the preceding proof of the arith^{l}. value of $\sin. 18°$, $(\cos. 18°)^2 = \frac{5+\sqrt{5}}{8}$ and $(\sin. 18°)^2 = \frac{3-\sqrt{5}}{8}$; $\therefore \cos. 2.\ 18°$, or, $\cos. 36° = (\cos. 18°)^2 - (\sin. 18°)^2 = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4} = \sin. 54°$, since $\cos. 36° = \sin. 54°$,

substitute this value of the $\sin. 54°$ in the forms page 46, for $\sin. 2A$, and there will result the above values for the sine and cosine of 27°.

Since, $\cos. 36° = \frac{\sqrt{5}+1}{4}$, $\sin. 36° = \frac{\sqrt{(10-2\sqrt{5})}}{4}$.

The subjoined Table contains the sines of arcs from 0 to 90 that differ by 9°, or in the French division of the circle, that differ by 10 degrees.

French Scale.	English Scale.	
sin. 10°, or, cos. 90°	sin. 9°, or, cos. 81°	$=\frac{1}{4}\sqrt{(3+\sqrt{5})}-\frac{1}{4}\sqrt{(5-\sqrt{5})}$
sin. 20, or, cos. 80	sin. 18, or, cos. 72	$=\frac{1}{4}(\sqrt{5}-1)$
sin. 30, or, cos. 70	sin. 27, or, cos. 63	$=\frac{1}{4}\sqrt{(5+\sqrt{5})}-\frac{1}{4}\sqrt{(3-\sqrt{5})}$
sin. 40, or, cos. 60	sin. 36, or, cos. 54	$=\frac{1}{4}\sqrt{(10-2\sqrt{5})}$
sin. 50, or, cos. 50	sin. 45, or, cos. 45	$=\frac{1}{\sqrt{2}}$
sin. 60, or, cos. 40	sin. 54, or, cos. 36	$=\frac{1}{4}(\sqrt{5}+1)$
sin. 70, or, cos. 30	sin. 63, or, cos. 27	$=\frac{1}{4}\sqrt{(5+\sqrt{5})}+\frac{1}{4}\sqrt{(3-\sqrt{5})}$
sin. 80, or, cos. 20	sin. 72, or, cos. 18	$=\frac{1}{4}\sqrt{(10+2\sqrt{5})}$
*sin. 90, or, cos. 10	sin. 81, or, cos. 9	$=\frac{1}{4}\sqrt{(3+\sqrt{5})}+\frac{1}{4}\sqrt{(5-\sqrt{5})}$

But the most general formula of verification is this, which is to be found in Euler's *Analysis Infinitorum*, page 201, vol. I. (Lausanne, 1748).

[5] sin. A + sin. $(36°-A)$ + sin. $(72°+A)$ = sin. $(36°+A)$ + sin. $(72°-A)$.

In order to prove this formula, we will use the numerical values of the cosines of 36° and 72°.

sin. $(36°+A)$ − sin. $(36°-A)$ = 2 . cos. 36° . sin. A =

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*There is a large Table of this kind in Cagnoli's *Trigonometry*, p. 58, &c. edit. 2. And, it is easy to see how additional formulæ may be obtained: for instance,

$$\sin. 15° = \sin.(45°-30°) = \sin. 45°(\cos. 30° - \sin. 30°)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right) = \frac{1}{2\sqrt{2}}(\sqrt{3}-1),$$

$$\text{Similarly, } \sin. 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

And these and like expressions, besides that utility which is pointed out in the text, may have a farther one, in the theory of Polygonals.

$\frac{\sqrt{5}+1}{2}$. sin. A and sin. $(72° + A) - \text{sin. } (72° - A) = 2 \text{ cos. } 72°$. sin. $A =$ $\frac{\sqrt{5}-1}{2}$. sin. A; subtract the latter equation from the former, and then we have, sin. $(36° + A) + \text{sin. } (72° - A) - \text{sin. } (72° + A) - \text{sin. } (36° - A) = \text{sin. } A$, which transposed is the equation [5].

If, in the above equation, we substitute $90° - A$ instead of A, then

$$\text{sin. } (90° - A) + \text{sin. } (A - 54°) + \text{sin. } (18° + A) = \text{sin. } (54° + A) + \text{sin. } (18° - A), \text{ or}$$

$$\text{sin. } (90° - A) = \text{sin. } (54° + A) + \text{sin. } (54° - A) - \text{sin. } (18° + A) - \text{sin. } (18° - A)$$

which is Legendre's mode of expressing the equation. But, it is plain from the mode by which the latter has been deduced, there is no real difference between the two formulæ, and, with regard to their application, it is quite indifferent, whether we adopt Euler's or Legendre's.

In using these formulæ, different values must be substituted for A, thus: in Euler's,

if $A = 9°$, then, sin. $9° + \text{sin. } 27° + \text{sin. } 81° = \text{sin. } 45° + \text{sin. } 63°$;
or, sin. $9° + \text{sin. } (3 \cdot 9°) + \text{sin. } (9 \cdot 9°) = \text{sin. } (5 \cdot 9°) + \text{sin. } (7 \cdot 9°)$

If in Legendre's formula we make $A = 81°$; then

$$\text{sin. } 9° = \text{sin. } 45° - \text{sin. } 27° - \text{sin. } 81° + \text{sin. } 63;$$
$$\text{or sin. } 9° + \text{sin. } 27° + \text{sin. } 81° = \text{sin. } 45° + \text{sin. } 63°,$$

the same as before; which proves what we have just asserted, (l. 11, &c.)

Again, if $A = 18°$, then, by [5] sin. $18° + \text{sin. } (36° - 18°) + \text{sin. } 90° = \text{sin. } (36° + 18°) + \text{sin. } (72° - 18°)$;

$$\text{or,} \quad 2 \text{ sin. } 18° + 1 = 2 \text{ sin. } 54°.$$

If $A = 10°$, then sin. $10° + \text{sin. } 26° + \text{sin. } 82° = \text{sin. } 46° + \text{sin. } 62°$.

EXAMPLE.

Take Sherwin's Tables, in which the natural sines of arcs are inserted: from these it appears, that

sin. 9°	= 1564345		
sin. 27	= 4539905	sin. 45°	= 7071068
sin. 81	= 9876883	sin. 63	= 8910065
	15981133		15981133
sin. 10°	= 1736482		
sin. 26	= 4383711	sin. 46°	= 7193398
sin. 82	= 9902681	sin. 62	= 8829476
	16022874		16022874

Here the numerical results of the two sums exactly agreeing, as they ought to do, according to the formula, we may conclude, almost with certainty, that in Sherwin's Tables the sines of 9, 27, 81, 45, 63, 10, 26, 46, 62 degrees are rightly computed.

These formulæ are most convenient for practice; but if, from the solution of an equation of three dimensions, as (s^{III}) p. 41, or, from that of an equation of five dimensions, as (s^{V}), the value of sin. $\frac{A}{3}$, or of sin. $\frac{A}{5}$ be computed, such value would become a means of ascertaining the accuracy of Trigonometrical Tables.

In 1610, Pitiscus published great Trigonometrical Tables, inserted in his *Mathematicus Thesaurus*, and, from the account of this Work, given in the Berlin Memoirs of 1786, page 24, it appears, that formulæ, which, in fact, are formulæ of verification, were employed by that mathematician: thus, in order to ascertain whether the chord of 30° had been rightly computed, he substitutes in the equation,

* $4x^2 - x^4 = c^2$ (x = chord of 30°, c = chord 60°)

* Pitiscus's notation is like Vieta's, given in page 53; the form of his equation is $4q - 1bq$ = square of chord of twice the arc. Another account of this Work of Pitiscus, and of other similar Works, published about the same time, and now very rare, is given in the 5th Volume of the *Memoirs of the Institute*.

instead of x the computed value, and he finds the resulting value of c to be 1000; as it ought to be, since the chord $60° = \text{radius}$: and accordingly he concludes x to have been rightly computed. Again, in order to ascertain that the chord of 10° is rightly computed, he substitues its computed value in an equation, such as chord $30° = 3$ chord $10° -$ (chord $10)^3$ [see form [s'''] page 49] and the chord 30° thence computed ought to agree with its value (x) previously ascertained to be right; and he pursues a similar course, in order to verify the computation of the chord of the fifth part of an arc.

Previously to quitting this part of our subject, we wish to employ the arithmetical values of the sines of 18° and of 36°, which have been just deduced, in proving that which may be announced as a geometrical property.

By the form p. 67, $\cos. 36° = \frac{\sqrt{5}+1}{4}$, and $\therefore \sin.^2 36° = \frac{5-\sqrt{5}}{8}$;

consequently, $4 \sin.^2 36° - 1 = \frac{3-\sqrt{5}}{2} = 4\sin.^2 18°$, see page 66,

or $(\text{chord } 72°)^2 - 1 = (\text{chord } 36°)^2$, but the chords of 72° and 36° are respectively the sides of an equilateral pentagon and decagon, inscribed in a circle. Hence, the square of the side of an equilateral pentagon inscribed in a circle, is equal to the square of the radius plus the square of the side of an equilateral decagon inscribed in the same circle.

Having now obtained methods of arithmetically computing the sines, cosines, &c. of angles, when the angles are expounded by a specific number of degrees, minutes, &c. we may proceed to apply our formulæ to express the various relations that subsist between the sides and angles of rectilinear triangles.

CHAP. V.

On the Resolution of the Cases of Rectilinear Triangles.—1st, When the Triangles are Right-angled.—2d, When Oblique.—Reasons for introducing different Solutions of the same Case. Examples, &c.

In a triangle there are 3 sides and 3 angles: any three of these being given, the remaining may be obtained. There is one exception to this, when the three quantities given are the 3 angles. The reason of the exception is this: take any triangle, then, externally or internally, other triangles may be formed with sides parallel to the sides of the proposed triangle, which triangles shall have the same angles, but greater sides or less sides: the magnitude of the sides therefore is independent of the angles, and consequently cannot be determined from them.

We will begin with the solution of the *cases of right-angled triangles.*

1st Case of right-angled triangles, in which two sides are given.

Here besides the right angle C, a, b, are given, and c, A, B are required.

Solution.

c determined.

1st. $c=\sqrt{(a^2+b^2)}$ Euclid 47. Book 1.

A and B determined.

2d. $\frac{\text{sin. } A}{\text{sin. } B}=\frac{a}{b}$ by Cor. 1 to Prob. 2.

but $A+B=\frac{\pi}{2}$ or 90° $\therefore$ sin. $B=$cos. A

$\therefore \frac{\text{sin. } A}{\text{cos. } A}$, or tan. $A=\frac{a}{b}$ (rad. $=1$), or

tan. $A=r.\frac{a}{b}$ and expressed in log^{ms}.

log. tan. $A=$ log. $r+$ log. $a-$log. b

$B=90°-A$

A being determined, c may; for

$$c=\frac{b.\text{sec. } A}{r}=\frac{br}{\text{cos. } A}$$

$\therefore$ log. $c=$log. $r+$log. $b-$log. cos. A.

Example.

$a=43$; $\therefore a^2=1849$

$b=55$; $\therefore b^2=3025$

$\therefore c^2=4874$

and $c=69{,}81$, &c.

Computation.

r the tabular radius $=10^{\text{ds}}$

log. $r=$.....10

log. $43=$.....1.6334685

11.6334685

log. $55=$.....1.7403627

$\therefore$ log. tan. $A=9.8931058$

$\therefore A=38°\ 1'\ 8''$

and $B=51\ 58\ 52$

Again, log $r=10$

log. $55=1.7403627$

11.7403627

log. cos. $38°\ 1'\ 8''=9.8964202$

log $c=1.8439425$

$\therefore c=69, 81$, &c. as before.

2d Case, in which the hypothenuse and one of the acute angles are given.

Here c, B, and $C = 90°$, are given, and a, b, A required.

Solution.

b determined.

$$\frac{\sin. B}{\sin. C} = \frac{b}{c} \text{ or } \frac{\sin. B}{1} = \frac{b}{c}$$

$$\therefore b = c . \sin. B = c . \frac{\sin. B}{r} \text{ (rad.} = r)$$

In logarithms,

$$\log. b = \log. c + \log. \sin. B - \log. r$$

$$A = 90° - B = 48°\ 48' \therefore \frac{\sin. 48°\ 48'}{\sin. 90°} = \frac{a}{c};$$

$$\text{or } a = c . \frac{\sin. 48°\ 48'}{r} \text{ and consequently, } a \text{ may be}$$

determined exactly as b has been; or thus, from b,

$$a = \sqrt{(c^2 - b^2)} = \sqrt{\{(c + b)(c - b)\}}.$$

$$\text{In Log}^{ms}. \log. a = \frac{1}{2}\{\log. (c+b) + \log. (c - b)\}$$

Example.

$c = 361 . 4$, $B = 41°\ 12'$

Computation.

log. 361 . 4	= 2.5579881
log. sin. 41° 12′	= 9. 8186807
log. b	= 2 . 3766688
$\therefore b$	= 238 . 05

3d Case, in which a side and the acute angle (which is not opposite to it) are given.

Here, b, A and $C = 90°$ are given, and a, c, B, are required.

Solution.

B determined.

$B = 90° - A$

a determined.

$$\frac{\sin. A}{\sin. B} = \frac{a}{b} \text{ [Cor. 1. Prob. 2]}$$

$$\therefore a = b . \frac{\sin. A}{\sin. B} = b . \frac{\sin. A}{\cos. A} = \frac{b \tan. A}{r};$$

$\therefore$ in logarithms,

$$\log. a = \log. b + \log. \tan. A - 10$$

c determined.

$$\frac{\sin. C}{\sin. B} = \frac{c}{b}; \therefore c = b . \frac{\sin. C}{\sin. B}$$

since $C = 90°$ and $\sin. B = \cos. A$

$$c = \frac{b}{\cos. A} = \frac{rb}{\cos. A} \text{ (rad.} = r)$$

In logarithms,

$$\log. c = 10 + \log. b - \log. \cos. A$$

Example.

$b = 31 . 76$ $A = 17°\ 12'\ 51''$

$\therefore B = 72°\ 47'\ 9''$.

Computation for a.

log. b or log. 31.76	= 1.5018805
log. tan. 17° 12′ 51″	= 9.4911132
log. a + 10	= 10.9929937
$\therefore a$	= 9.8399

Computation for c.

log. 10^{10}	= 10
log. 31 . 76	= 1.5018805
10 + log. b .	= 11.5018805
log. cos. 17° 12′ 51″	= 9.9800967
log. c	= 1.5217838
$\therefore c$	= 33.249.

We now proceed to the *Cases of Oblique-angled Triangles.*

First Case, in which two angles and a side opposite to one of the angles are given.

Here, A, B, a, are given, and b, c, C are required.

Solution.

C determined.

$A+B+C=180^\circ \therefore C=180^\circ-(A+B)$

b determined.

by Cor. 1, Prob. 2.

$$\frac{\sin. B}{\sin. A}=\frac{b}{a}; \therefore b=a.\frac{\sin. B}{\sin. A}$$

In logarithms,

$\log. b=\log. a+\log. \sin. B-\log. \sin. A$

The side c is similarly determined.

Example.

$A=41^\circ\ 13'\ 22'', B=71^\circ\ 19'\ 5'', a=55$

$\therefore C=180^\circ-(112^\circ\ 32'\ 27'')=67^\circ\ 27'\ 33''$

$\log. a$ or $\log. 55 = 1.7403627$

$\log. \sin. B$

or $\log. \sin 71^\circ\ 19'\ 5''=9.9764927$

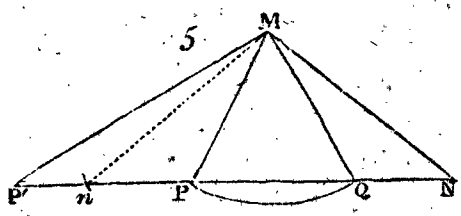
$\log. a+\log. \sin. B=11.7168554$

$\log. \sin. A= 9.8188779$

$\therefore \log. b = 1.8979775$

$\therefore b = 79.063$

Second Case of oblique-angled triangles, in which, two sides and an angle opposite to one of the sides are given.



Here, a, b, B are given, and A, C, c are required.

Solution.

A determined.

$$\frac{\sin. A}{\sin. B}=\frac{a}{b} \therefore \sin. A=\sin. B\,\frac{a}{b}$$

In logarithms,

$\log. \sin. A=\log. \sin. B+\log. a-\log. b$

This case may be ambiguous, or will admit of two solutions, when $a>b$, and B is acute: for, let $MN=a$, $MP=b$, $\angle MNP=B$ take $Mn=MN$, then MP ($<Mn$) falls between N, n,

Example 1. (ambiguous)

$a=178.3$, $b=145$, $B=41^\circ\ 10'$

Computation for A

$\log. \sin. B$

or $\log. \sin. 41^\circ\ 10'=9.8183919$

$\log. a$ or $\log. 178.3 = 2.2511513$

12.0695432

$\log. b$ or $\log. 145 = 2.1613680$

$\log. \sin. A = 9.9081752$

$\therefore$

N, n, and another line $M2$, also between N, n, can be taken equal to it; ∴ the triangle may be MNP, or $MN2$, and the angle A may be either $MP2$ or its supplement $M2N$; but, if b or MP be > a or MN, MP would fall beyond N and n, as MP' does, and no other line equal to it can be drawn between P' and N: in this case, A has one value only. If B be obtuse, A cannot; therefore here also the case is not ambiguous.

A and B being known, $C = 180° - (A + B)$ is known.

c determined.

$$\therefore \frac{\sin. C}{\sin. B} = \frac{c}{b} \therefore c = b \cdot \frac{\sin. C}{\sin. B};$$

or c may directly, that is without the intervention of the process for finding A, be determined from this expression

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ whence,}$$

$$c = a \cos. B \pm \sqrt{(b^2 - a^2 \sin.^2 B)}$$

∴ $A = 54° 2' 22''$
and $C = 84\ 47\ 38$
or $A = 125\ 57\ 38$
and $C = 12\ 52\ 22$

Example 2. (not ambiguous)
$a = 145$, $b = 178.3$, $B = 41° 10'$

log. sin. $41° 10' = 9.8183919$
log. $145 = 2.1613680$
11.9797599
log. $178.3 = 2.2511513$
log. sin. $A = 9.7286086$
∴ $A = 32° 21' 54''$

In this instance the supplement of A cannot belong to the case proposed.

Computation of c in 1st Example.

log. $145 = 2.1613680$
log. sin. $84° 47' 38''$ 9.9982047
12.1595727
log. sin. $41° 10' = 9.8183919$
log. $c = 2.3411808$
∴ $c = 219.37$

Third Case, in which two sides and the included angle are given.

Here, a, b, C are given, and A, B, c are required.

By Cor. 1, Prob. 2. $\frac{a}{b} = \frac{\sin. A}{\sin. B}$

$$\therefore \frac{a}{b} - 1 = \frac{\sin. A}{\sin. B} - 1$$

$$\text{or } \frac{a-b}{b} = \frac{\sin. A - \sin. B}{\sin. B};$$

$$\text{similarly } \frac{a+b}{b} = \frac{\sin. A + \sin. B}{\sin. B}$$

∴

Example.

$a = 562$, $b = 320$, $C = 128° 4'$.

Computation.

$a - b = 242$
$a + b = 882$
$A + B = 180° - (128° 4') = 51° 56'$
$\frac{A+B}{2} = 25° 58'$

log.

$\therefore$

$$\frac{a-b}{a+b}=\frac{\sin. A-\sin. B}{\sin. A+\sin. B}=\frac{\tan.\frac{A-B}{2}}{\tan.\frac{A+B}{2}}$$

by [*f*] page 27

compute $\therefore$ tan. $\frac{(A-B)}{2}$ from this expression; and $\frac{A-B}{2}$ is known from the Trigonometrical Tables, and since

$A+B=180°-C$ we shall have A and B, for if

$$\frac{A+B}{2}=s, \text{ and } \frac{A-B}{2}=d$$

then, $A=s+d$, and $B=s-d$

c determined.

$$\frac{\sin. C}{\sin. A}=\frac{c}{a} \therefore c=a\cdot\frac{\sin. C}{\sin. A}.$$

In logarithms, log. c = log. a + log. sin. C − log. sin. A, or c may directly be found, thus:

$$\cos. C=\frac{a^2+b^2-c^2}{2ab}$$

$$\therefore c=\sqrt{(a^2+b^2-2ab\cos. C)}$$

which form however is not suited to logarithmic computation.

log. tan. 25° 58′ = 9.6875402
log. 242 = 2.3838154
12.0713556
log. 882 = 2.9454686

log. tan. $\frac{A-B}{2}$ = 9.1258870

$\therefore \frac{A-B}{2}$ = 7° 36′ 40″

$\frac{A+B}{2}$ = 25° 58′

$\therefore A$ = 33° 34′ 40″
B = 18° 21′ 20″

Computation of *c*.

log. 562 = 2.7497363
log. sin. 128° 4′ } or log. sin. 51 56 } = 9.8961369
12.6458732
log. sin. 33° 34′ 40″ = 9.7427789
log. c = 2.9030943
$\therefore c$ = 800.01.

The above is a complete solution of the case, in which two sides and the included angle are the quantities given. But, the analytic art is required to furnish, besides merely adequate, commodious and concise, solutions. And, of this latter character, are the solutions which have been given of the third case by Dr. Maskelyne, in the Introduction to Taylor's *Logarithms*, and by Legendre, in his *Trigonometry*, p. 369, 4th edit. These solutions we now proceed to explain.

Let $a > b$: find in the Tables an angle θ, such that tan. $\theta = r\frac{a}{b}$ and from this logarithmic expression:

$$\log.\ \tan.\ \theta = 10 + \log.\ a - \log.\ b \ldots\ldots\ldots\ldots\ldots [\alpha]$$

in this case, since $a > b$, θ is $> 45°$, for when $\theta = 45°$, tan. $\theta = r$.

Now, since tan. $\theta = r.\frac{a}{b}$, tan. $\theta \mp r = r\left(\frac{a}{b} \mp 1\right) = r\frac{a \mp b}{b}$;

$$\therefore \frac{\tan.\ \theta - r}{\tan.\ \theta + r} = \frac{a-b}{a+b} = \frac{\tan.\left(\frac{A-B}{2}\right)}{\tan.\left(\frac{A+B}{2}\right)}$$

but by Prob. 4, page 29, &c. tan. $(\theta - 45°) = r\left(\frac{\tan.\ \theta - r}{\tan.\ \theta + r}\right)$,

hence, r tan. $\left(\frac{A-B}{2}\right) = \tan.\left(\frac{A+B}{2}\right)$. tan. $(\theta - 45°) \ldots\ldots\ldots [\beta]$

consequently, since θ is known from the expression $[\alpha]$ tan. $\frac{A-B}{2}$ may be computed; and thence, by means of the Trigonometrical Tables, $\frac{A-B}{2}$ is known, and A and B may be determined as in the former case.

Solution of the preceding Example by this method.

Computation of θ by the formula $[\alpha]$

$$\begin{aligned} 10. + \log.\ 562 &= 12.7497363 \\ \log.\ 320 &= 2.5051500 \\ \therefore \log.\ \tan.\ \theta &= 10.2445863 \quad \therefore \theta = 60°\ 20'\ 35'' \\ \theta - 45° &= 15°\ 20'\ 35'' \end{aligned}$$

Computation of $\frac{A-B}{2}$ by the formula $[\beta]$

$$\begin{aligned} \log.\ \tan.\ 15°\ 20'\ 35'' &= 9.4383476 \\ \log.\ \tan.\ \frac{A+B}{2}, \text{ or } \log.\ \tan.\ 25°\ 58' &= 9.6875402 \\ \therefore 10 + \log.\ \tan.\ \frac{A-B}{2} &= 19.1258876 \end{aligned}$$

$$\therefore \frac{A-B}{2} = 7^\circ\ 36'\ 40''$$

and since $\frac{A+B}{2} = 25^\circ\ 58'$

$A = 33^\circ\ 34'\ 40''$ and $B = 18^\circ\ 21'\ 20''$

the same result as was obtained by the preceding method, and obtained in practice, almost with equal facility, even when it is necessary to take from the Tables the logarithms of a and b.

The demonstration of this method is not more concise than that of the preceding; but, the rule and the connected computation are, and, especially, in those cases in which the logarithms of a and b should happen to be given; for then θ would immediately be determined from the form $[\alpha]$, and $\frac{A-B}{2}$ from the form $[\beta]$, so that the whole of the rule would be expressed by the two forms $[\alpha]$, $[\beta]$.

The above method of determining the angles A, B, is the same as that which is contained in the fourth Proposition of Robert Simpson's *Trigonometry*, p. 486, of his Elements of Euclid, 6th edition: and, in substance, is the same as the method given by Dr. Maskelyne, p. 36. Introduction to Taylor's *Logarithms*: the sole difference of the two methods is in the expression: instead of the formula,

$$r \cdot \text{tan.}\left(\frac{A-B}{2}\right) = \text{tan.}\left(\frac{A+B}{2}\right) \text{tan.}\,(\theta - 45^\circ)$$

Dr. Maskelyne directs us to employ

$$r \cdot \text{tan.}\left(\frac{A-B}{2}\right) = \text{co-tan.}\ \frac{C}{2} \cdot \text{tan.}\,(\theta - 45^\circ)$$

but, since $A+B+C = 180^\circ$, $\frac{C}{2} = \left(90^\circ - \frac{A+B}{2}\right)$

consequently, $\text{co-tan.}\ \frac{C}{2} = \text{tan.}\left(\frac{A+B}{2}\right)$

Two methods of computing the side c, have been already given, one, from A and B previously determined; the other, independent of such determination; the latter method, however, is not adapted to logarithmic computation; but, it may be by the introduction of an angle, called a subsidiary angle, such as θ is in the preceding demonstration, thus: from page 75,

$$c = \sqrt{(a^2+b^2-2ab.\cos.C)}$$
$$= \sqrt{(a^2-2ab+b^2+2ab-2ab.\cos.C}$$
$$= \sqrt{\{(a-b)^2+2ab(1-\cos.C)\}}$$
$$= (a-b)\sqrt{\left(1+\frac{2ab}{(a-b)^2}\text{ver. sin. }C\right)}$$

Assume $\frac{4ab}{(a-b)^2}\cdot\frac{\text{ver. sin. }C}{2}=\tan.^2\theta$ and $\therefore 1+\frac{4ab}{(a-b)^2}\cdot\frac{\text{ver. sin. }C}{2}$ $=1+\tan.^2\theta=\sec.^2\theta$;

consequently,

$$c=(a-b).\sec.\theta, \text{ or} = \frac{a-b}{\cos.\theta}\quad(\text{rad.}=1)$$

$$\text{or } c=(a-b)\frac{\sec.\theta}{r}, \text{ or} = \frac{(a-b)r}{\cos.\theta}\quad(\text{rad.}=r)$$

and in logarithms,

$$\log. c=\log.(a-b)+\log.\sec.\theta-10, \text{ or} =\log.(a-b)+10-\log.\cos.\theta.$$

This agrees with Dr. Maskelyne's determination of c, given page 36, Introduction to Taylor's *Logarithms*: and the sole difference in the process is, that instead of

$$\frac{\text{ver sin. }C}{2},\text{ he uses, }\sin.^2\frac{C}{2};$$

which two values, as it appears from p. 37, are equal.

Example of the computation of c : a and b, being 562 and 320, and $C=128°\ 4'$.

$$2\log.\tan.\theta-10=\log.2+\log.a+\log.b+\log.\text{ver. sin. }C-2\log.(a-b)$$

now log. 2 = 0.3010300
log. 562 = 2.7497363
log. 320 = 2.5051500
log. ver. sin. 128° 4′ = 10.2085966
15.7645129
2 log. 242 = 4.7676308
10.9968821

$\therefore$ log tan. θ = 10.4984410; $\therefore$ log. sec. θ = 10.5192823
$\therefore$ log. c = 10.5192823 + 2.3838154 − 10 = 2.9030977
$\therefore$ c = 800.01

If we make the value of c equal to

$$\sqrt{(a^2+2ab+b^2-2ab-2ab\cos.C)}$$

we shall have $c = (a + b)\sqrt{\left(1 - \frac{2ab}{(a+b)^2}(1 + \cos. C)\right)}$

and, if we assume $\frac{4ab}{(a+b)^2} \cdot \left(\frac{1+\cos. C}{2}\right) = \sin^2. \theta$,

there will result, $c = (a + b) . \cos. \theta$

$1+\cos. C$ is the versed sine of the supplement of C, which Mr. Mendoza, in his valuable Tables on *Nautical Astronomy*, calls the * *suversed sine*; hence the rule for the solution algebraically expressed, is log. sin. θ

$$= \tfrac{1}{2}\{10 + \log.2 + \log. \text{suver. sin. } C + \log. a + \log. b - 2 \log. (a+b)\}$$

$$\log. c = \log. (a+b) + \log. \cos. \theta - 10.$$

Of the preceding solutions of the third Case, one alone, as it has been remarked, is, in strictness, sufficient: the others have been added, for the sake of rendering, in certain cases, the computation more expeditious. And, when a specific instance is presented, it will not be difficult to determine which method of solution it is, that ought to be adopted. If, for example, the side opposite the included angle be alone required, we ought to compute it by the method of p. 78, l. 11, avoiding, as unnecessary, the calculation of the angles (A, B,) at the base.

Fourth Case, in which the three sides are given.

Here, a, b, c, are given, and A, B, C, are required.

First Solution.

By Prob. 2, page 21, sin. A =

$$\frac{2}{bc}\sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)\right\}}$$

---

* In these Tables the log. suversed sine = log. $\frac{1+\cos.}{2}$.

and, in logarithms, putting $\frac{a+b+c}{2} = S$, and instead of sin. A, $\frac{\text{sin. } A}{r}$,

$$\text{log. sin. } A - 10 = \text{log. } 2 + \tfrac{1}{2}\{\text{log. } S + \text{log. } (S-a) + \text{log. } (S-b) + \text{log. } (S-c)\} - \text{log. } b - \text{log. } c$$

and similarly, log. sin. C is expressed by the same form, substituting in the negative part, instead of $-\text{log. } c$, $-\text{log. } a$, so is also

log. sin. B, substituting instead of $-\text{log. } b$, $-$ log. a.

Second Solution.

By Prob. 2, page 21.

$$1 - \text{cos. } A = \frac{(a+b-c)(a+c-b)}{2bc}$$

$$= \frac{4}{2bc}\left\{\left(\frac{a+b+c}{2} - c\right)\left(\frac{a+b+c}{2} - b\right)\right\}$$

But $1 - \text{cos. } A = 2 \cdot \text{sin.}^2 \frac{A}{2}$, [page 37.] hence, introducing the radius r,

$$\frac{\text{sin.}^2 \cdot \frac{A}{2}}{r^2} = \frac{(S-b)(S-c)}{bc}$$

In logarithms,

$$2 \text{ log. sin. } \frac{A}{2} = 20 + \text{log. } (S-b) + \text{log. } (S-c) - \text{log. } b - \text{log. } c$$

and similarly,

$$2 \text{ log. sin. } \frac{C}{2} = 20 + \text{log. } (S-a) + \text{log. } (S-b) - \text{log. } a - \text{log. } b.$$

Third Solution.

By Prob. 2, page 21,

$$1 + \cos. A = \frac{(a+b+c)(c+b-a)}{2bc} = 2\frac{S.(S-a)}{bc}.$$

But $1+\cos. A = 1-\cos.(\pi - A) = 2\sin.^2\frac{\pi - A}{2} = 2\sin.^2\left(\frac{\pi}{2} - \frac{A}{2}\right)$

$= 2.\cos.^2\frac{A}{2}$. Hence,

$$\cos.^2.\frac{A}{2} = r^2 \times \frac{S.(S-a)}{bc}.$$

In logarithms,

$$2\log.\cos.\frac{A}{2} = 20 + \log. S + \log.(S-a) - \log. b - \log. c$$

and similarly,

$$2\log.\cos.\frac{C}{2} = 20 + \log. S + \log.(S-c) - \log. a - \log. b.$$

Fourth Solution.

Divide the expression for the $\sin.^2\frac{A}{2}$ by that for $\cos.^2\frac{A}{2}$, and since the tangent is equal to the radius multiplied into a fraction of which the numerator is the sine and the denominator the cosine, we have, $\tan.^2\frac{A}{2} = r^2 \times \frac{(S-b)(S-c)}{S.(S-a)}$.

In logarithms,

$$2\log.\tan.\frac{A}{2} = 20 + \log.(S-b) + \log.(S-c) - \log. S - \log.(S-a)$$

and similarly,

$$2\log.\tan.\frac{C}{2} = 20 + \log.(S-b) + \log.(S-a) - \log. S - \log.(S-c)$$

Example by the first method of solution.

$a=33$ log.$=1.5185139$
$b=42.6$ log.$=1.6294096$
$c=53.6$ log.$=1.7291648$

$\therefore$ log. a+log. b=3.1479235 [C]*
log. b+log. c=3.3585744 [A]
log. a+log. c=3.2476787 [B]

$\frac{a+b+c}{2}$ $=64.6$ log.$=1.8102325$

$\frac{a+b+c}{2}-a=31.6$ log.$=1.4996871$

$\frac{a+b+c}{2}-b=22$ log.$=1.3424227$

$\frac{a+b+c}{2}-c=11$ log.$=1.0413927$

2 | 5.6937350 |
2.8468675 = log. area : Cor. 2. Prob. 2.
10.3010300 $\therefore$ area=702.858

13 . 1478975.

Hence, (see the formula of solution) if we subtract from this logarithm (13.1478975) the values [C], [A], [B], we shall have, respectively, the log. sines of the angles C, A and B:

13.1478975	13.1478975	13.1478975
[C] 3.1479235	[A] 3.3585744	[B] 3.2476787
9.9999740=log. sin. C	9.7893231=log. sin. A	9.9002188=log. sin. B
$\therefore$ $C=89°\ 22'\ 21''$	$A=37°\ 59'\ 53''$	$B=52°\ 37'\ 46''\frac{9}{16}$
or 89 22 22	$B=52\ 37\ 46\frac{9}{16}$	
to 89 22 25	$\therefore$ $C=89\ 22\ 20\frac{7}{16}$	
	180 0 0	

In this case, if we had not determined A and B, the value of C would have been doubtful to the extent of 4": consequently, for the finding the exact value of C, in this instance, the first method is not proper.

* [C] [A] [B] are merely marks of reference.

By the second method; the same Example.

Angle C computed.		Angle A computed.	
$\log.(S-a) =$	1.4996871	$\log.(S-b) =$	1.3424227
$\log.(S-b) =$	1.3424227	$\log.(S-c) =$	1.0413927
(20 added)	22.8421098	(20 added)	22.3838154
$[C]$ or $\log.a+\log.b$	3.1479235	$[A]$	3.3585744
$2\log.\sin.\frac{C}{2} =$	19.6941863	$2\log.\sin.\frac{A}{2} =$	19.0252410
$\log.\sin.\frac{C}{2} =$	9.8470931	$\log.\sin.\frac{A}{2} =$	9.5126205
$\therefore C = 2(44^\circ\ 41'\ 10''\frac{4}{21})$		$\therefore A = 2(18^\circ\ 59'\ 56''\frac{20}{61})$	
$= 89\ 22\ 20\frac{8}{21}$		$= 37\ 59\ 52\frac{50}{61}$	

By this method the angle C is determined at once with great accuracy.

By the third method.

Angle C computed.		Angle A computed.	
$\log. S =$	1.8102325	$\log. S =$	1.8102325
$\log.(S-c) =$	1.0413927	$\log.(S-a) =$	1.4996871
(20 added)	22.8516252	(20 added)	23.3099196
$[C]$	3.1479235	$[A]$	3.3585744
	19.7037017		19.9513452
$\log.\cos.\frac{C}{2} =$	9.8518508,5	$\log.\cos.\frac{A}{2} =$	9.9756726
$\therefore C = 2(44^\circ\ 41'\ 10''\frac{45}{210})$		$\therefore A = 2(18^\circ\ 59'\ 56''.5)$	
$= 89\ 22\ 20\frac{9}{21}$		$= 37\ 59\ 53$	

As C, in these two last methods, is determined to a great degree of exactness, the value of B is not computed from the formula, but it may be had by subtracting $A+C$ from 180°.

By the fourth method.

Angle C computed.

$\log.(S-b) = 1.3424227$
$\log.(S-a) = 1.4996871$
(20 added) 22.8421098 (c)

Again,

$\log. S = 1.8102325$
$\log.(S-c) = 1.0413927$
2.8516252 (d)

$\therefore (c)-(d) = 19.9904846$

$\therefore \log. \tan. \frac{C}{2} = 9.9952423$

$\therefore C = 2\,(44^\circ\ 41'\ 10''\frac{2}{41})$
$= 89\ 22\ 20\frac{82}{41}$

Angle A computed.

$\log.(S-b) = 1.3424227$
$\log.(S-c) = 1.0413927$
(20 added) 22.3838154 (a)

Again,

$\log. S = 1.8102325$
$\log.(S-a) = 1.4996871$
3.3099196 (b)

$\therefore (a)-(b) = 19.0738958$

$\therefore \log. \tan. \frac{A}{2} = 9.5369479$

$\therefore A = 2\,(18^\circ\ 59'\ 56''.5)$
$= 37\ 59\ 53.$

As far as instances prove, any one of these three latter methods may be used for determining the angle C, and angles nearly of the same magnitude; and, it is of no material consequence which it is that is used. The first method is plainly, from a mere comparison of results, insufficient to give exactly the value of such an angle as C is: and we need not go through the labour of the arithmetical computation in order to ascertain its insufficiency: for, if we perceive, that the square of the side such as c, is nearly equal to the sum of the squares of the other two sides, we shall know that the value of C does not differ much from 90°.*

It may now be worth the while to enquire, more minutely, why, since compendium of calculation is a desirable object, several methods of solution have been given.

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* This 4th case of oblique triangles is commonly (see Robert Simpson's *Euclid*, page 488; Ludlam, page 220,) solved by means of this proposition. The sum of the two sides of a triangle is to the base as the difference of the segments of the base is to the difference of the sides; but the demonstration of this proportion, since the case is otherwise more conveniently solved, is purposely omitted.

Now, each of the preceding methods is adapted to logarithmic computation, and each, in an analytical point of view affords a complete solution. One solution, therefore, would have been sufficient, and one alone given, if the same applied, with equal convenience and equal numerical accuracy, to all instances; but the fact is otherwise. If an example were proposed in which the angle A should be nearly 90°, as C is in the former Example: the log. sin. A might be deduced from the first solution; but, to such logarithmic value, there would not, in the Tables, correspond a precise value of the angle A: for instance, if the numerical value of log. sin. A should be 9.9999998, A might equal (by the Tables) either 89° 56′ 19″, or 89° 57′ 8″, or any angle intermediate of these two angles. The reason of this is, the very small variation of the sine of an angle nearly equal to 90°. And, this small variation is apparent from the mere inspection of the Geometrical diagram, in which two contiguous sines should be drawn to two arcs each nearly equal to a quadrant; or, analytically, it may be thus shewn. Let A be an arc nearly $=90°$; let it be increased by a small quantity (1″ for instance), then by the formula [1], p. 23, making $B = 1''$,

$$\text{sin.}\,(A + 1'') = \text{sin.}\,A\,.\,\text{cos.}\,1'' + \text{cos}\,A\,.\,\text{sin.}\,1''.$$

Subtract sin. A from each side of the equation, then

$$\text{sin.}\,(A + 1'') - \text{sin.}\,A = \text{sin.}\,A\,(\text{cos.}\,1'' - 1) + \text{cos.}\,A\,\text{sin.}\,1'',$$

(by p. 37, l. 19,) $$= -\text{sin.}\,A\,.\,2\,\text{sin.}^2\,\frac{1''}{2} + \text{cos.}\,A\,\text{sin.}\,1''.$$

Now, sin. $A \times 2\,\text{sin.}^2\,\frac{1''}{2}$, may, from the smallness of the factor $2\,\text{sin.}^2\,\frac{1''}{2}$, be neglected; and, accordingly,

$$\text{sin.}\,(A + 1'') - \text{sin.}\,A = \text{cos.}\,A\,.\,\text{sin.}\,1'',\ \text{nearly};$$

therefore, the difference of two contiguous sines, or what has been called, the variation of the sine, varies nearly as cos. A; and the cos. A is when $A=90°$, nearly, a very small quantity relatively to its other values, in which A is of a mean value.

It must not, however, be unnoticed, that the want of precision in the determination of the angle is partly owing to the con-

struction of the Logarithmic and Trigonometrical Tables. The Tables referred to, and in common use *, are computed to seven places of figures; but, if we had Tables † computed to a greater number of places, to double the number, for instance, then the logarithmic sines of all angles between 89° 56′ 18″, and 89° 57′ 9″, would not be expressed, as they are in Tables now in use, by the same figures. In such circumstances, we should obtain conclusions very little remote from the truth; but then, such Tables would be extremely incommodious for use, and would, in all common cases, give results to a degree of accuracy quite superfluous and useless. Moreover, such Tables, even in the extreme cases which we have mentioned, are not essentially necessary: since their use can be superseded, by abandoning the first method of solution, and recurring either to the 2d, 3d, or 4th method.

When the angle (*A*) sought then is nearly = 90°, the first method must not be used, but one of the latter methods, in which either the sine, cosine or tangent of *half* the angle is determined; and, in such an extreme case, it is a matter of indifference whether, instead of the first method, we substitute the 2d, or 3d, or 4th. But, in other cases, it is not a matter of indifference: for since, as it has been shewn, the variation or the increment of the sine is as the cosine, and of the cosine as the sine, these two variations are equal at 45°, but beyond 45°, up to 90°, that of the sine is less, and that of the cosine greater; and, the contrary happens between 45° and 0; consequently we have this Rule.

if the angle sought be < 90°, use the second method;
if.....> 90°, use the third method.

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* Sherwin's 8vo. Hutton's 8vo. Taylor's 4to.

† In Vlacq's Tables, published at Gouda, 1633, we have

Arcs.			Log. Sines.
89°	56′	10″	9.9999997300
		20	97530
		40	97958
		50	98154
89	57	0	98346
		10	98525

The 4th method may be used, and commodiously, for all values of the angles sought from 0 up to angles nearly = 180°: when, however, the angle (A) is nearly = 180°, tan. $\frac{A}{2}$ which is nearly tan. 90°, is very large, and its variations, (which are as the square of the secant*) are also very large and irregular. If, therefore, we use Sherwin's Tables, which are computed for every minute only of the quadrant, the logarithms corresponding to the seconds, taken out by proportional parts, will not be exact: for in working by proportional parts, it is supposed, if the difference between the logarithmic tangents of two arcs differing by 60 seconds be d, that the difference between the logarithmic tangents of the first arc, and of another arc, that differs from it only by n seconds is $\frac{n}{60}d$: now, this is not true for arcs nearly equal to 90°; and an Example will most simply shew it: by Sherwin's Tables,

log. tan. 89° 30′ = 12.0591416
log. tan. 89 29 = 12.0449004.....[2]

log. diff. corresponding to 60″ = 142412
diff. corresponding to 30″ = 71206.....[4]

∴ by Rule, log. tan. 89° 29′ 30″ ([2]+[4]) = 12.0520210
whereas true log. tan. 89° 29′ 30″, by Taylor's Log. = 12.0519626

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* For by the formula, p. 30,

$$\text{tan. } (A+1'') = \frac{\text{tan. } A + \text{tan. } 1''}{1 - \text{tan. } A . \text{tan. } 1''}.$$

Subtract tan. A from each side, and

$$\text{tan. } (A+1'') - \text{tan. } A = \frac{\text{tan. } 1'' + \text{tan.}^2 A \text{ tan. } 1''}{1 - \text{tan. } A . \text{ tan. } 1''}$$

by expanding and neglecting terms involving tan.^{2} 1″, $= (\text{tan. } 1'' + \text{tan.}^2 A . \text{tan. } 1'')(1 + \text{tan. } A \text{ tan. } 1'')$

$= \text{tan. } 1''(1 + \text{tan.}^2 A)$ nearly;

∴ since tan. 1″ is an assigned quantity,

tan. $(A + 1'')$ − tan. $A \propto 1 + \text{tan.}^2 A \propto \text{sec.}^2 A$.

Again,

log. tan. 89° 50′ = 12.5362727
log. tan. 89 49 = 12.4948797

log. diff. corr^{s}. to 60″ = 413930
∴ diff. corr^{s}. to 6″ = 41393

∴ by the Rule log. tan. 89° 49′ 6″ = 12.4990190
whereas the true log. tan. 89° 49′ 6″, by Taylor's Log^{ms}. = 12.498845

In these instances, the log. tangent, determined by the proportional parts, is too large, which it plainly must be; for, the logarithmic increment of the tangent increasing as the arc does, that is, the increment during the last 30″ being greater than the increment during the first 30″, if we take half the whole increment for the increment due to the first 30″, or one-tenth of the whole increment, for the increment due to the first 6″, we plainly take quantities too large. This same reason would, it is true, hold against calculating logarithmic tangents of any arcs by proportional parts, if the values of logarithmic tangents were exactly put down in Tables; but (we speak of the Tables in ordinary use) the values are expressed by seven places only of figures; and, as far as seven places, the irregularities in the successive differences of the logarithmic tangents of arcs that are of some mean value, between 0 and 90°, do not appear; thus, by Sherwin's Tables,

log. tan. 44° 30′ = 9.9924197
log. tan. 44 29 = 9.9921670

log. diff. corresponding to 60″ = 2527
∴ diff. corresponding to 30 = 12635

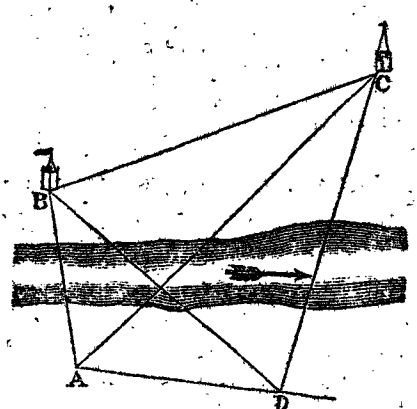
∴ by the Rule log. tan. 44° 29′ 30″ = 9.9909335
and the true log. tan. by Taylor's Tables = 9.9922934.

It appears then, from the assigned reason, and by the instances given, that an angle nearly 90° cannot exactly be found from its logarithmic tangent. The determination of the angle by means of proportional parts will be wrong in seconds by Sherwin's Tables; and will be wrong in the parts of seconds by Taylor's Tables. From the whole of what has been said then, it appears that in computing the values of angles, two inconveniences may occur either when the successive logarithmic numbers are too nearly alike, as in the case of sines of angles nearly 90°, or too widely

different, as in the case of the tangents of angles nearly equal to 90°. It is the business of the Analyst to provide formulæ, by which these inconveniences may be remedied or avoided; and hence have arisen the different methods for attaining, apparently the same end.

Before we entirely relinquish this digression, we wish to observe, that, although the log. sine or log. tangent of the angle *A* may be determined exactly either by the first or the fourth method, yet, if it should be very small, its value cannot, with sufficient exactness, be determined by the Tables in common use. For, very small angles cannot be exactly found from their logarithmic sines and tangents; not exactly in seconds, by Sherwin's Tables, nor exactly, in parts of seconds, by Taylor's Tables; and therefore, as great exactness may be required, and is commonly required, in those cases, in which a very small angle is to be determined, the Tables are not to be used. They are to be superseded by a peculiar computation, of which, without demonstration, Dr. Maskelyne has given the rule in his Introduction to Taylor's *Logarithms*, p. 17 and 22. This rule and similar rules will be stated and demonstrated in a subsequent part of this Work, when the analytical series for the sine and tangent of an arc are deduced.

To the several cases of the solution of oblique triangles, examples have been given, but, merely arithmetical examples; it may be proper therefore, to subjoin a feigned case of practice and observation, in order to shew, more plainly, the use and application of the formulæ of solution.



An observer at *A* wishes to determine his distance from two inaccessible objects *B*, *C*, and also the distance *BC*, of the same objects.

The observer takes a new station D, and measures the distance AD; suppose it to equal 1763 yards: at A and D, by means of proper instruments, he makes the following observations:

$$\text{at } A \begin{pmatrix} BAC = 45^\circ\ 1'\ 3'' \\ CAD = 30^\circ\ 0'\ 2'' \end{pmatrix}, \text{ at } D \begin{pmatrix} BDC = 36^\circ\ 15'\ 5'' \\ BDA = 33^\circ\ 7'\ 40'' \end{pmatrix}$$

consequently,

$CDA = 69^\circ\ 22'\ 45''$, $ACD = 180^\circ - (ADC + CAD) = 80^\circ\ 37'\ 13''$

$BAD = 75^\circ\ 1'\ 5''$, $ABD = 180^\circ - (BAD + BDA) = 71^\circ\ 51'\ 15''$

AB determined.

By 1st Case of oblique triangles, p. 73.

$$\frac{AB}{AD} = \frac{\text{sin. } ADB}{\text{sin. } ABD};$$

$\therefore$ log. AB = log. AD + log. sin. ADB − log. sin. ABD.

log. 1763 = 3.2462523
log. sin. 33° 7′ 40″ = 9.7375966
12.9838489
log. sin. 71° 51′ 15″ = 9.9778456
log. AB = 3.0060033
$\therefore$ AB = 1014.

AC determined.

By 1st Case of oblique triangles,

$$\frac{AC}{AD} = \frac{\text{sin. } ADC}{\text{sin. } ACD};$$

$\therefore$ log. AC = log. AD + log. sin. ADC − log. sin. ACD.

log. 1763 = 3.2462523
log. sin. 69° 22′ 45″ = 9.9712441
13.2174964
log. sin. 80° 37′ 13″ = 9.9941543
log. AC = 3.2233421
$\therefore$ AC = 1672.4

CB determined.

By formula of computation given, page 78,

$$\frac{\tan.^2 \theta}{r} = \frac{2ab}{(a-b)^2} \text{ ver. sin. } C$$

and, log. tan. θ

$$= \tfrac{1}{2} \left\{ \begin{matrix} 10 + \text{log. } 2 + \text{log. } a + \text{log. } b \\ + \text{log. versin. } C - 2\text{log. } (a-b) \end{matrix} \right\}$$

and CB or $c = \frac{r}{\cos. \theta} \times (a-b)$

and

log. c = 10 + log. $(a-b)$ − log. cos. θ

here, $a = AC = 1672.4$, $b = AB = 1014$

$C = BAC = 45^\circ\ 1'\ 3''$, $a - b = 658.4$

10 + log. 2 = 10.3010300
log. 1672.4 = 3.2233421
log. 1014 = 3.0060033
log. versin. 45° 1′ 3″ = 9.4670294
25.9974048
2 log. 658.4 = 5.6369796
20.3604252
$\therefore$ log. tan. θ = 10.1801466
and log. cos. θ = 9.7412271 [a]
$\therefore$ since
10 + log. 658.4 = 12.8184898 [b]
log. c = 3.0772627 [b] − [a]
and c = 1194.7.

The last part of this Example (the determination of CB) belongs to the third case of oblique-angled triangles, in which two sides and the included angle are given; but, since the angles at the base, that is, the angles ABC, ACB, are not required, the method of solution given (see p. 75,) was not adopted: BC, indeed, may be computed, and, according to the common practice of calculators, would be computed, by first determining the angles ABC, ACB from the form

$$\text{tan.}\left(\frac{ABC-ACB}{2}\right)=\frac{AC-AB}{AC+AB}\cdot\text{tan.}\left(\frac{ABC+ACB}{2}\right)$$

and then BC from this expression, $BC=AB.\frac{\text{sin. }BAC}{\text{sin. }ACB}$; but it is plain, that the computation, without being at all more exact, would be longer than that by which BC has been already determined,

The last part of the Example will also serve to illustrate the use of that solution of the third case, which was given in page 75: for, the logarithms of AC and AB being determined in the previous part, we have immediately

log. tan. $\theta = 10 + (3.2233421 - 3.0060033) = 10.2173388$;
whence $\theta = 58°\ 46'\ 23''$, and $\theta - 45° = 13°\ 46'\ 28''$;
consequently, since log. tan. $13°\ 46'\ 23''.8 = 9.3893876$,

log. tan. $\left(\frac{ABC+ACB}{2}\right)$ or co-tan. $22°\ 30'\ 31''.5 = 10.3835881$;

$$\therefore \text{log. tan. } \frac{ABC-ACB}{2} = 9.7729757$$

$$\text{and } \frac{ABC-ACB}{2} = 30°\ 39'\ 48''$$

$$\text{but } \frac{ABC+ACB}{2} = 67°\ 29'\ 28''.5$$

$$\text{consequently, } ABC = \overline{98°\ 9'\ 16''.5}$$
$$ACB = 36°\ 49'\ 40''.5$$

The process had been somewhat more tedious if we had found these angles by the formula given in page 75, in the solution of the third case, for then we must have computed

$AC - AB$, $AC + AB$, and have taken out their logarithms. This instance, and the remark on it, have been introduced to shew, not that the common and general solution is insufficient, but that other solutions may conveniently, that is, with some gain of expedition, be introduced. The Student, however, who shall peruse this Treatise in order to be initiated into Trigonometry, is advised, in the first perusal, to attend solely to the general solutions, and to postpone to a time of leisure and of acquired knowledge, the consideration of the methods that are either more expeditious, or are adapted to particular exigencies*.

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* With the view of rendering every thing as easy as possible to the Student, separate investigations of the cases of right-angled triangles, have been made to precede those of oblique-angled triangles; but, considered generally, the former cases are really included in the latter, and their solutions comprehended within the general solutions; we will shew this in two instances: suppose in the third case of oblique triangles, that the included angle is a right angle, then by the solution,

$$\text{tan.}\left(\frac{A-B}{2}\right) = \text{tan.}\left(\frac{A+B}{2}\right).\frac{a-b}{a+b}, \text{ but } A+B+C = 180^\circ \text{ and } C = 90^\circ$$

$$\therefore \frac{A+B}{2} = 45^\circ;$$

consequently, $\text{tan.}\frac{A-B}{2} = \text{tan. } 45^\circ.\frac{a-b}{a+b} = \frac{a-b}{a+b}$, since tan. $45^\circ = 1$;

but $A = 90^\circ - B$;

$$\therefore \frac{A-B}{2} = 45^\circ - B; \therefore \text{tan.}\left(\frac{A-B}{2}\right) = \text{tan. } (45^\circ - B) = \text{(by Prob. 4,}$$

Cor. 2, p. 30.) $\frac{1 - \text{tan. } B}{1 + \text{tan. } B}$;

hence $\frac{a-b}{a+b} = \frac{1-\text{tan. } B}{1+\text{tan. } B}$ and tan. $B = \frac{b}{a}$ the same solution, in fact, as was given, page 71, in the solution of the first case of right-angled triangles.

Suppose next, in the same instance, we employ the form used in the fourth case of oblique-angled triangles, thus: $1+$

We will terminate this Chapter with one or two Problems, the solutions of which may be deduced, almost immediately, from the preceding formulæ.

1. In a triangle of which the sides are a, b, c, and the angles opposite, A, B, C, it is required to determine C, when a, b, and $A-B$ are given.

By the formula of p. 75,

$$\tan.\left(\frac{A-B}{2}\right)=\tan.\left(\frac{A+B}{2}\right)\times\frac{a-b}{a+b}=\tan.\left(90^\circ-\frac{C}{2}\right)\times\frac{a-b}{a+b}$$

$$=\text{(see p. 10,) }\cot.\frac{C}{2}\times\frac{a-b}{a+b}=\text{(see p. 11,) }\frac{1}{\tan.\frac{C}{2}}\times\frac{a-b}{a+b};$$

$$\therefore\ \tan.\frac{C}{2}=\frac{a-b}{a+b}\times\cot.\frac{A-B}{2}.$$

2. Given, A, B, C, and $a\pm b$; required a and b.

If the upper sign be used, then $a-b=(a+b)\times\tan.\frac{C}{2}\times\tan.\frac{A-B}{2}$

If the lower sign be used, then, $a+b=(a-b).\cot.\frac{C}{2}\cot.\frac{A-B}{2}$

and in each case, by adding and subtracting, we obtain $2a$, and $2b$, and thence a and b.

3. Given A, a and $b+c$; required b, c, &c.

By the third formula of solution, p. 81,

$$1+\cos.A=\frac{\left(\frac{a+b+c}{2}\times\frac{b+c-a}{2}\right)}{bc},\ \text{but}\ \frac{b+c+a}{2}\times\frac{b+c-a}{2}$$

$$=\frac{b^2+c^2-a^2+2bc}{4}=\frac{b^2+bc}{2};\ \text{since}\ c^2=a^2+b^2:$$

$$\text{hence, } 1+\cos.A=\frac{b^2+bc}{bc}=\frac{b}{c}+1\ \text{and}\ \cos.A=\frac{b}{c},$$

and $\therefore\ c=\frac{b}{\cos.A}$ the same result as that which was obtained, page 7:

so that it is plain the variety of cases might have been diminished, but not without a considerable loss of simplicity and facility.

$$2^2 \cdot \cos.^2 \frac{A}{2} = \frac{(a+b+c)(b+c-a)}{bc}.$$

Hence bc is known, let it $= p$, and let $b + c = s$;

then $b + c = s$

$bc = p$;

$\therefore$ (by the solution of a quadratic equation)

$$b = \frac{s}{2} + \sqrt{\left(\frac{s^2}{4} - p\right)}$$

$$c = \frac{s}{2} - \sqrt{\left(\frac{s^2}{4} - p\right)}.$$

4. Given $a + b + c$, the sum of the sides, the area of the triangle, and the angle A; it is required to find the side a (see Newton, *Arith. Univ.* Prob. 8.)

By Prob. 2, p. 22, l. 1,

$$1 + \cos. A = \frac{(b+c)^2 - a^2}{2bc} = \frac{(a+b+c-a)^2 - a^2}{2bc}.$$

Again, the area $(K) = \sin. A \times \frac{bc}{2}$;

$$\therefore 1 + \cos. A = \frac{(a+b+c)^2 - 2a(a+b+c)}{4K} \times \sin. A;$$

$$\therefore a = \tfrac{1}{2}(a+b+c) - \frac{2K}{a+b+c} \times \frac{1 + \cos. A}{\sin. A}$$

$$= \tfrac{1}{2}(a+b+c) - \frac{2K}{a+b+c} \times \cot. \frac{A}{2};$$

since (Cor. 5. p. 28.) $\dfrac{\sin. A}{1 + \cos. A} = \tan. \dfrac{A}{2} = \dfrac{1}{\cot. \dfrac{A}{2}}$.

The formulæ of Trigonometry have now been applied to the resolution of rectilinear triangles; the original object, for which the science was invented. And, it is to be observed, such application is the most easy, and is of very extensive practical utility. In the next Chapter we will continue to apply, still

farther, the preceding formulæ. The first instances will shew their utility in expediting Arithmetical computation: the latter, selected from Works and Writings on *Physical Astronomy*, will shew their utility in subjects of great importance and of arduous investigation. In this last application, the original object of the science, the Properties of Triangles, seems entirely to be lost sight of, and the Trigonometrical analysis is peculiarly and almost solely useful, because it confers precision and power on Mathematical language.

CHAP. VI.

Instances of the Utility of Trigonometrical Formulæ.

1. It is required to compute the logarithm of $a \pm b$,

$$a \pm b = a\left(1 \pm \frac{b}{a}\right).$$

Assume, for the upper sign, $\frac{b}{a} = \tan.^2 \theta$, for the lower, $\frac{b}{a} = \sin.^2 \theta$

then log. $(a + b) =$

log. a + log. $(1 + \tan.^2 \theta) =$ log. $a + 2$ log. sec. $\theta - 20$,

and log. $(a - b) =$

log. a + log. $(1 - \sin.^2 \theta) =$ log. $a + 2$ log. cos. $\theta - 20$.

In the application of this method, a and b are not supposed to be numbers: for, then, the simple way would be to add them and to take the logarithm of the sum; but a and b are compounded quantities, formed of the sines and cosines of angles: thus, in finding the Moon's distance from a star,

versin. dist. = versin. $(d' - d)$ + cos. d. cos. d'. versin. A

(d. d' being the declinations, and A the difference of the right ascensions) a is = versin. $(d' - d)$ and b = cos. d. cos. d' versin. A, and $\tan.^2 \theta$ is assumed =

$$\frac{\text{cos. } d. \text{ cos. } d' \text{ versin. } A}{\text{versin. } (d' - d)}, \text{ or, } = \frac{\text{cos. } d. \text{ cos. } d' \text{ sin.}^2 \frac{A}{2}}{\text{sin.}^2 \frac{d' - d}{2}};$$

and then, by the process above stated, the form is adapted to logarithmic computation.

M. Sejour solves this instance somewhat differently (see his *Traité Analyt.* tom. I. p. 103)

Thus, let $a + b = y$,

then, a sin. $45° + b$ cos. $45° = y$. sin. $45°$ (1)

Assume $\tan. z = \frac{b}{a}$; $\therefore \frac{\sin. z}{\cos. z} = \frac{b}{a}$, and

$$a \sin. z - b \cos. z = 0 \qquad (2).$$

Eliminate a from the equations (1) and (2), and

$$b (\sin. z \cos. 45° + \cos. z . \sin. 45°) = y \sin. 45° \sin. z,$$

$$\text{or } b . \sin. (45° + z) = y \sin. 45° \sin. z,$$

$$\text{and } \therefore y, \text{ or } a + b = \frac{b . \sin. (45° + z)}{\sin. 45° . \sin. z}.$$

Again, in order to compute the logarithm of $a + b + c$, since $a + b + c = y + c$, make $\tan. z' = \frac{c}{y}$

$$= \frac{c}{b} \cdot \frac{\sin. 45° . \sin. z}{\sin. (45° + z)}, \text{ then, as before,}$$

$$a + b + c = c . \frac{\sin. (45° + z')}{\sin. 45° . \sin. z'},$$

and so on for $a + b + c + d$, &c.

2. It is required to compute the sine of an angle; for instance, the sine of 3°.

$$\text{Sin. } 3° = \sin. (2° + 1°) = \sin. 2° \cos. 1° + \cos. 2° \sin. 1°,$$

$$= \sin. 2° \cos. 1° \left(1 + \frac{\sin. 1° \cos. 2°}{\cos. 1° \sin. 2°}\right)$$

$$= \sin. 2° \cos. 1° \left(1 + \frac{\tan. 1°}{\tan. 2°}\right).$$

Assume $\therefore \tan.^2 \theta = \frac{\tan. 1°}{\tan. 2°}$, and we shall have, in a logarithmic form,

$$\log. \sin. 3° = \log. \sin. 2° + \log. \cos. 1° + 2 \log. \sec. \theta - 30.$$

3. Let it be required to compute a quantity P, such that $P = (1 + e')(1 + e'')(1 + e''') \times$ &c.; the law of the formation of e', e'', e''', &c. being

$$e' = \frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}, e'' = \frac{1-\sqrt{(1-e'^2)}}{1+\sqrt{(1-e'^2)}}, e''' = \frac{1-\sqrt{(1-e''^2)}}{1+\sqrt{(1-e''^2)}}.$$

The computation is conveniently effected, thus, put $e = \text{sin. } \theta$;

$\therefore \sqrt{(1-e^2)} = \text{cos. } \theta$, and $e' = \frac{1-\text{cos. } \theta}{1+\text{cos. } \theta} = $ [page 28] $\text{tan.}^2 \frac{\theta}{2}$

$$\text{and, } 1+e' = 1+\text{tan.}^2 \frac{\theta}{2} = \text{sec.}^2 \frac{\theta}{2}.$$

Again, put $e' = \text{sin. } \theta'$;

$$\therefore e'' = \frac{1-\text{cos. } \theta'}{1+\text{cos. } \theta'} = \text{tan.}^2 \frac{\theta'}{2} \text{ and } 1+e'' = \text{sec.}^2 \frac{\theta'}{2}.$$

Again, put $e'' = \text{sin. } \theta''$; $\therefore$

$$e'' = \frac{1-\text{cos. } \theta''}{1+\text{cos. } \theta''} = \text{tan.}^2 \frac{\theta''}{2}, \text{ and } 1+e''' = \text{sec.}^2 \frac{\theta''}{2}, \text{ \&c.}$$

$$\text{Hence } P = \text{sec.}^2 \frac{\theta}{2} \cdot \text{sec.}^2 \frac{\theta'}{2} \cdot \text{sec.}^2 \frac{\theta''}{2}, \text{ \&c.}$$

and log. $P =$ [supplying the tabular radius]

$$2\left(\text{log. sec. } \frac{\theta}{2} + \text{log. sec. } \frac{\theta'}{2} + \text{log. sec. } \frac{\theta''}{2} + \text{\&c.}\right)$$

$$- 2(10 + 10 + 10 + \text{\&c.})$$

4. * Required the integrals of the differential expressions, $d\theta \cdot \text{sin. } \theta \cdot \text{cos. } \theta$; $d\theta \cdot \text{cos.}^2 \theta$; $d\theta \cdot \text{sin.}^2 \theta$; $d\theta \text{ cos.}^3 \theta$; $d\theta \cdot \text{cos. } \theta \text{ cos. } n\theta$; $\frac{d\theta}{\text{cos.}^4 \theta}$.

In order to integrate these expressions, it is necessary to premise (see Simpson's *Fluxions*, Vol. I, p. 165.) θ being any arc, that

$$d\theta = \frac{d(\text{sin. } \theta)}{\text{cos. } \theta}, \text{ or } = -\frac{d(\text{cos. } \theta)}{\text{sin. } \theta}, \text{ or } = \frac{d(\text{tan. } \theta)}{\text{sec.}^2 \theta}.$$

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* This, in other words and symbols, is to require the fluents of the fluxionary expressions $\theta \cdot$, $\text{sin. } \theta \text{ cos. } \theta \cdot$; $\theta \cdot \text{cos.}^2 \theta$, &c.

1st. $\int d\theta \,.\, \cos.\,\theta \,.\, \sin.\,\theta.$

By the formula of p. 36,

$$\cos.\,\theta \,.\, \sin.\,\theta = \frac{1}{2} \,.\, \sin.\,2\theta;$$

$$\therefore \int d\theta \,.\, \cos.\,\theta \,.\, \sin.\,\theta = \frac{1}{2}\int d\theta \,.\, \sin.\,2\theta = \frac{1}{4}\int 2d\theta \,.\, \sin.\,2\theta$$

$$= -\frac{1}{4}\cos.\,2\theta + c \quad (c = \text{correction}).$$

2d. $\int d\theta \,.\, \cos.^2\theta.$

By the formula of p. 53,

$$\cos.^2\theta = \frac{1}{2}(1 + \cos.\,2\theta);$$

$$\therefore \int d\theta \,.\, \cos.^2\theta = \int\frac{d\theta}{2} + \frac{1}{2}\int d\theta \,.\, \cos.\,2\theta$$

$$= \int\frac{d\theta}{2} + \frac{1}{4}\int 2d\theta \,.\, \cos.\,2\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin.\,2\theta + c.$$

3d. $\int d\theta.\ \sin.^2\theta.$

By the formula of p. 54,

$$\sin.^2\theta = \frac{1}{2}(1 - \cos.\,2\theta);$$

$$\therefore \int d\theta \,.\, \sin.^2\theta = \int\frac{d\theta}{2} - \frac{1}{2}\int d\theta \,.\, \cos.\,2\theta$$

$$= \int\frac{d\theta}{2} - \frac{1}{4}\int 2d\theta \,.\, \cos.\,2\theta$$

$$= \frac{\theta}{2} - \frac{1}{4}\sin.\,2\theta + c.$$

4th. $\int d\theta \,.\, \cos.^3\theta.$

By the formula of p. 53,

$$\cos.^3\theta = \frac{1}{4}\cos.\,3\theta + \frac{3}{4}\cos.\,\theta;$$

$$\therefore \int d\theta \cos.^3\theta = \frac{1}{4}\int d\theta . \cos. 3\theta + \frac{3}{4}\int d\theta . \cos.\theta$$

$$= \frac{1}{3.4}\int 3\, d\theta . \cos. 3\theta + \frac{3}{4}\sin.\theta$$

$$= \frac{1}{3.4}. \sin. 3\theta + \frac{3}{4}\sin.\theta + c.$$

5th. $\int d\theta . \cos.\theta . \cos. n\theta.$

By the formula of p. 39, l. 19,

$$\cos.\theta . \cos. n\theta = \frac{1}{2}\left\{\cos.(n-1)\theta + \cos.(n+1)\theta\right\};$$

$$\therefore \int d\theta . \cos. n\theta = \frac{1}{2}\int d\theta . \cos.(n-1)\theta + \frac{1}{2}\int d\theta . \cos.(n+1)\theta$$

$$= \frac{1}{2.(n-1)}\int (n-1)\, d\theta . \cos.(n-1)\theta$$

$$+ \frac{1}{2.(n+1)}\int (n+1)\, d\theta \cos.(n+1)\theta =$$

$$\frac{1}{2.(n-1)}\sin.(n-1)\theta + \frac{1}{2(n+1)}\sin.(n+1)\theta + c.$$

6th. $\int \frac{d\theta}{\cos.^4\theta}$.

By p. 12, l. 3, $\frac{1}{\cos.^4\theta} = \sec.^4\theta = (\sec.^2\theta)^2 = (1+\tan.^2\theta)^2$;

$$\therefore \int \frac{d\theta}{\cos.^4\theta} = \int d\theta (1+\tan.^2\theta)^2 = \int d\theta (1+\tan.^2\theta) \times (1+\tan.^2\theta)$$

$$= \int \left\{d(\tan.\theta)(1+\tan.^2\theta)\right\} = \tan.\theta + \frac{1}{3}\tan.^3\theta + c.$$

5. If $\frac{1}{y} = \frac{1-\varepsilon\cos.\varsigma}{1-\varepsilon^2}$, it is required to express $\frac{1}{y^2}$, $\frac{1}{y^3}$, &c. in terms involving the cosines of multiples of the arc ς; see Mayer's *Theory of the Moon*, p. 14.

$$\frac{1}{y} = \frac{1 - \varepsilon \cdot \cos. \varsigma}{1 - \varepsilon^2} = 1 + \varepsilon^2 - \varepsilon \cdot \cos. \varsigma,$$

rejecting * the terms involving ε^3, ε^4, &c.

$$\therefore \frac{1}{y^2} = (1+\varepsilon^2)^2 - 2 \cdot (1+\varepsilon^2) \varepsilon \cdot \cos. \varsigma + \varepsilon^2 \cdot \cos.^2 \varsigma,$$

$$= 1 + 2\varepsilon^2 - 2\varepsilon \cdot \cos. \varsigma + \varepsilon^2 \left\{ \frac{1}{2} + \frac{\cos. 2\varsigma}{2} \right\}$$

$$= 1 + \frac{5\varepsilon^2}{2} - 2\varepsilon \cdot \cos. \varsigma + \frac{\varepsilon^2}{2} \cos. 2\varsigma.$$

Again,

$$\frac{1}{y^3} = (1 + \varepsilon^2)^3 - 3\varepsilon \cdot (1 + \varepsilon^2)^2 \cos. \varsigma + 3\varepsilon^2 (1 + \varepsilon^2) \cos.^2 \varsigma$$

$$= 1 + 3\varepsilon^2 - 3\varepsilon \cdot \cos. \varsigma + 3\varepsilon^2 \left\{ \frac{1}{2} + \frac{\cos. 2\varsigma}{2} \right\}$$

$$= 1 + \frac{9\varepsilon^2}{2} - 3\varepsilon \cdot \cos. \varsigma + \frac{3\varepsilon^2}{2} \cos. 2\varsigma,$$

and generally

$$\frac{1}{y^n} = (1+\varepsilon^2)^n - n(1+\epsilon^2)^{n-1} \varepsilon \cdot \cos. \varsigma + n \cdot \frac{n-1}{2} (1+\varepsilon^2)^{n-2} \varepsilon^2 \cdot \cos.^2 \varsigma$$

$$= 1 + n\varepsilon^2 - n\varepsilon \cdot \cos. \varsigma + n \cdot \frac{n-1}{2} \varepsilon^2 \left\{ \frac{1}{2} + \frac{\cos. 2\varsigma}{2} \right\}$$

$$= 1 + \frac{n \cdot n+3}{4} \varepsilon^2 - n\varepsilon \cdot \cos. \varsigma + \frac{n \cdot n-1}{4} \varepsilon^2 \cos. 2\varsigma.$$

In this instance, it is plain, if all terms involving higher powers of ε than that of the square be rejected, that no higher multiple of ς than 2ς can be introduced. The quantity ς is the mean anomaly and y is the radius vector.

6. If $s = \sqrt{(f^2 - 2fr \cdot \cos. t + r^2)}$, it is required to express $\frac{1}{s^3}$ by a series of terms involving the cosines of the mul-

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* The rejection of the terms involving ϵ^3, ϵ^4, &c. is not, as it is plain, essential to the illustration of the use of Trigonometrical formulæ, but, we have given the instance as Mayer has, and as all similar instances in Physical Astronomy will be circumstanced, in which ϵ denotes the eccentricity of an orbit.

tiples of the arc t. (See *Mem. Acad. des Sciences*, 1754, p. 538. *Clairaut sur l'orbite apparent du Soleil*).

Let $2fr\cos.t - r^2 = a$, then

$s = \sqrt{(f^2 - a)}$, and by the binomial theorem,

$$\frac{1}{s^3} = \frac{1}{f^3} + \frac{3a}{2f^5} + \frac{15a^2}{8f^7} + \frac{35a^3}{16f^9} + \frac{315}{128}\frac{a^4}{f^{11}} + \&c.$$

But, $a^2 = r^4 - 4fr^3.\cos.t + 4f^2r^2.\cos.^2 t$,

(by Prob. 6,) $= r^4 - 4fr^3.\cos.t + 2f^2r^2(1 + \cos.2t)$

$= r^4 + 2f^2r^2 - 4fr^3\cos.t + 2f^2r^2.\cos.2t.$

Again,

$a^3 = -r^6 + 6fr^5\cos.t - 12f^2r^4\cos.^2 t + 8f^3r^3\cos.^3 t$

$= -r^6 - 6f^2r^4 + (6fr^5 + 6f^3r^3)\cos.t - 6f^2r^4.\cos.2t + 2f^3r^3\cos.3t.$ (Prob. 7, [e'''])

Substituting these values in the series for $\frac{1}{s^3}$, there results

$$\frac{1}{s^3} = \frac{1}{f^3} + \frac{9r^2}{4f^5} + \frac{225r^4}{64f^7} + \left(\frac{3r}{f^4} + \frac{45r^3}{8f^6}\right)\cos.t + \left(\frac{15r^2}{4f^5} + \frac{105r^4}{16f^7}\right)\cos.2t + \frac{35r^3}{8f^6}\cos.3t.$$

In this instance, r is the radius of the Earth's orbit, and f that of Jupiter's, and since $\frac{r}{f}$ is a small fraction ($=.19245$) the terms involving $\frac{r^5}{f^8}$, $\frac{r^6}{f^9}$, &c. are rejected.

The quantity $\frac{1}{s^3}$ is a factor in a term dependent, in the theory of a planet disturbed in its orbit, on the disturbing force; and, the object of the above resolution is to resolve the term into a series such as

$$A^{(0)} + A^{(1)}.\cos.t + A^{(2)}\cos.2t + \&c.$$

which is easily effected when $\frac{r}{f}$ is a small fraction, and may be, but not without artifice, when $\frac{r}{f}$ is nearly equal 1, that is, when the radii of the disturbing and disturbed body are nearly equal. (See *Phil. Trans.* 1804, pp. 265, &c. *Mem. Acad.* 1754, p. 545.)

7. If the terms of the series

$$A^{\circ} + A^{(1)} \cos. t + A^{(2)} \cos. 2t + \&c.$$

be multiplied by cos. mt, it is required so to transform the terms that the series shall preserve its original form.

By the formula *[d]* of p. 26,

$$\cos. nt \,.\, \cos. mt = \frac{1}{2} \cdot \{ \cos. (m-n)t + \cos. (m+n)t \}$$

Hence,

$$\cos. t \,.\, \cos. mt = \frac{1}{2} \{ \cos. (m-1)t + \cos. (m+1)t \}$$

$$\cos. 2t \,.\, \cos. mt = \frac{1}{2} \{ \cos. (m-2)t + \cos. (m+2)t \}$$

&c.

which values being substituted in

$$A^{(0)} \,.\, \cos. mt + A^{(1)} \cos. t \,.\, \cos. mt + A^{(2)} \cos. 2t \,.\, \cos. mt + \&c.$$

and the terms properly arranged, what is required will be done. (See Laplace, *Mem. Acad.* 1785, pp. 54, &c. and *Mecanique Celeste*, p. 263.)

8. Required the sin. t, cos. t, sin. $2t$, &c.

when $t = nv - \frac{2e}{m} (1 - n) \sin. mv$;

the coefficient $\frac{2e}{m} (1 - n)$ being a very small quantity.

(See *Acad. des Sciences*, 1754, p. 539: also for similar instances, *Acad. des Sciences*, 1745, p. 343, and Clairaut's *Theory of the Moon*, p. 20.)

By the formula [1] p. 23,

$$\text{sin. } t = \text{sin. } nv \,.\, \text{cos. } \left\{\frac{2e}{m}(1-n)\,\text{sin. } mv\right\} - \text{cos. } nv \,.\, \text{sin. } \left\{\frac{2e}{m}(1-n)\,\text{sin. } mv\right\}.$$

Now the quantity* $\frac{2e}{m}(1-n)$ sin. mv, by the hypothesis, is very small; therefore its cosine is nearly $= 1$, and its sine is nearly the arc which it is supposed to represent: consequently,

$$\text{sin. } t = \text{sin. } nv - \frac{2e}{m}(1-n)\,\text{sin. } mv \,.\, \text{cos. } nv,$$

[by (*b*) p. 26,] $= \text{sin. } nv - \frac{e}{m}(1-n)\left\{\text{sin.}(n+m)v - \text{sin.}(n-m)v\right\}$

Again,

$$\text{cos. } t = \text{cos. } nv + \frac{2e}{m}(1-n)\,\text{sin. } mv \,.\, \text{sin. } nv$$

[by (*e*) p. 26,] $= \text{cos. } nv + \frac{e}{m}(1-n)\left\{\text{cos. } (n-m)v - \text{cos. } (n+m)v\right\}$

and, by a like process, may sin. $2t$, cos. $2t$, &c. be deduced.

9. Required the value of

$$\dagger \text{ sin. } vfQ \text{ cos. } v \,.\, dv - \text{cos. } vfQ \text{ sin. } v \,.\, dv,$$

when Q is represented by a series of terms

$$a \,.\, \text{cos. } nv + b \,.\, \text{cos. } mv + \text{&c.}$$

(See Clairaut, *Acad. des Sciences*, 1745, p. 341, also his *Theorie de la Lune*, edit. 2. p. 9. Lalande, *Acad. des Sciences*, 1760, p. 313.

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* As in the former instance, (see Note, p. 101,) so in this, the smallness of the quantity $\frac{2e}{m}(1-n)$ is, in no wise, essential to the illustration of the use of the Trigonometrical formulæ.

† dv is the differential of v answering to $\dot{v}$ the fluxion of v.

Laplace, *Mec. Celeste*, p. 241. Thomas Simpson's *Tracts*, pp. 92, &c. Cousin's, *Physical Astronomy*, pp. 23, &c. Vince's, *Astronomy*, Vol. II. pp. 168, &c.

If we substitute the first term $a \cdot \cos. nv$ of the series in the expression, then

$$\sin. v \int Q \cos. v \cdot dv = a \cdot \sin. v \int \cos. v \cdot \cos. nv \cdot dv$$

[form (*d*) p. 26,] $= a \sin. v \frac{1}{2} \int \{\cos. (n-1)v + \cos. (n+1)v\} dv$

(p. 99,) $= \frac{a}{2} \sin. v \left\{\frac{\sin. (n-1)v}{n-1} + \frac{\sin. (n+1)v}{n+1} + C\right\}$

where C, the correction, will $= 0$, if the integral $= 0$ when $v = 0$.

Again,

$$\cos. v \int Q \sin. v \cdot dv = a \cos. v \int \sin. v \cdot \cos. nv \cdot dv,$$

(form [*b*] p. 26,) $= a \cos. v \frac{1}{2} \int \{\sin. (n+1)v - \sin. (n-1)v\} dv$

(p. 98, l. 19,) $= a \cos. v \left\{\frac{\cos. (n-1)v}{2(n-1)} - \frac{\cos. (n+1)v}{2(n+1)} + C'\right\}$

in which, according to the preceding hypothesis of the correction,

$$0 = \frac{1}{2 \cdot (n-1)} - \frac{1}{2 \cdot (n+1)} + C',$$

and $$C' = -\frac{1}{n^2 - 1}.$$

Hence, combining the two parts of the expression,

$$\sin. v \int Q \cdot \cos. nv \cdot dv - \cos. v \int Q \sin. nv \cdot dv =$$

$$\frac{a}{2(n-1)} \{\sin. (n-1)v \cdot \sin. v - \cos. (n-1)v \cdot \cos. v\}$$

$$+ \frac{a}{2 \cdot (n+1)} \{\cos. (n+1)v \cdot \cos. v + \sin. (n+1)v \sin. v\}$$

$$+ \frac{a \cdot \cos. v}{n^2 - 1} = \text{(by the forms [2] and [4]) pp. 25, 26,}$$

$$-\frac{a\,.\,\cos.\,n\,v}{2(n-1)}+\frac{a\,.\,\cos.\,n\,v}{2\,.\,(n+1)}+\frac{a\,.\,\cos.\,v}{n^2-1}=$$

$$\frac{a\,.\,\cos.\,v}{n^2-1}-\frac{a\,.\,\cos.\,n\,v}{n^2-1}.$$

If, instead of $a\,.\,\cos.\,n\,v$, we had substituted $b\,.\,\cos.\,m\,v$, for Q, the resulting value of the integral would have been

$$\frac{b\,.\,\cos.\,v}{m^2-1}-\frac{b\,.\,\cos.\,m\,v}{m^2-1}.$$

Hence, the whole integral, when Q is represented by the series $a\,.\,\cos.\,n\,v+b\,.\,\cos.\,m\,v+$ &c. is equal

$$\left\{\frac{a}{n^2-1}+\frac{b}{m^2-1}+\&c.\right\}\cos.\,v$$

$$-\left(\frac{a\,.\,\cos.\,n\,v}{n^2-1}+\frac{b\,.\,\cos.\,m\,v}{m^2-1}+\&c.\right).$$

10. In the Lunar Theory, one of the *equations* (that of the *evection*) is represented by $-2\,r\,.\,\sin.\,(2\,X-Z)$, r being supposed to be of a mean constant value: if r should vary and proportionally to $1-3\,c\,.\,\cos.\,Y$, what is the new *equation* that would thence arise? (See Vince's *Astronomy*, vol. II. p. 53.)

Let m be the mean value of r, then

$$\text{the } \textit{evection} = -2m\,(1-3c\,.\,\cos.\,Y)\,\sin.\,(2X-Z)$$

$$= -2m\,\sin.\,(2X-Z)+6\,m\,c\,\cos.\,Y\,.\,\sin.\,(2X-Z).$$

Therefore, since the first term denotes the mean value, the second, that is, $6\,m\,c\,\cos.\,Y\,\sin.\,(2X-Z)$ denotes the variation from that mean value, or the *new equation*.

Now, $6\,m\,c\,.\,\cos.\,Y\,.\,\sin.\,(2X-Z) =$ (by form [*a*] p. 26,)

$$3\,m\,c\left\{\sin.\,(2X-Z+Y)+\sin.\,(2X-Z-Y)\right\}$$

which is the form of the new *equation*, in which, the angles $2X-Z+Y$, $2X-Z-Y$, are technically denominated the *arguments*, and $3\,m\,c\,(=2'\,1'')$ is the *coefficient*.

The Student, perhaps, may now be inclined to believe that the formulæ demonstrated in the preceding pages, are not entirely without their use, nor invented and shewn as mere specimens of analytical dexterity. The instances, indeed, have been, almost all, taken from Tracts on Physical Astronomy; and it is, therefore, on the assumption of the utility of that science, that the Trigonometrical analysis has been affirmed to be useful.

We will proceed to apply the formulæ of Trigonometry to the resolution of certain numerical equations; an application less extensive than the preceding, and of more doubtful utility.

CHAP. VII.

On the Solution of certain Numerical Equations by means of Trigonometrical Formulæ and Tables.

In this application of the Trigonometrical Analysis, the utility, whatever it be, relates to the expedition and convenience of the resolution of the equations, and not to any thing novel or curious in its principle and method; moreover, the expedition and conciseness of the resolution depends not on any essential and real abridgment, but on that sort and kind of abridgment which registered computations or tables afford: for instance, we shewed (page 61) that a cubic equation, such as

$$\text{sin. } 3A = 3 \text{ sin. } A - 4 \text{ sin.}^3 A$$

when solved by approximation, might be used for the computation of sines, or the construction of Tables. Reversely, the Tables constructed may be used for the resolution of similar cubic equations; again, we shewed (page 61) that an equation of 5 dimensions, such as

$$2 \text{ sin. } 5A, \text{ or } x = 5x - 5x^3 + x^5$$

solved by approximation, might be made subservient to the construction of Tables. Reversely, Tables constructed either by the approximate solution of equations, or by other methods of approximation, may be employed in the numerical resolution of similar equations. De Moivre solved equations of the third and fifth degree by the cosines of the third part and fifth of an arc; and Vieta divided an arc into three and five equal parts, by equations of the third and fifth degree. This is sufficient, perhaps, to explain the real principle of the solution of equations by Trigonometrical Tables. The convenience or expedition of the method, as we have already said, is of the same nature as the expedition of computation by logarithms. If we do not avail ourselves of computed Tables, the whole process of the solution of a cubic equation, for instance, will be tedious; we must employ some method

of approximation; now, if Trigonometrical Tables are employed, the process is short and easy, but only so, because the most laborious part of it is already done for us. In these cases, we avail ourselves of the registered computations of preceding mathematicians.

Solution of a Cubic Equation.

It may be proper, however, to shew more in detail the method of solving equations by the aid of Trigonometrical Tables.

If we take the equations [c'''] and [s'''] pp. 39, 41, supply a radius r, and put c and s for cos. $3A$, and sin. $3A$, respectively, and x for cos. A and sin. A, then

$$cr^2 = 4x^3 - 3r^2x, \text{ or } x^3 - \frac{3r^2}{4}x - \frac{cr^2}{4} = 0 \ldots\ldots\ldots\ldots [1]$$

$$\text{and } sr^2 = 3r^2x - 4x^3, \text{ or, } x^3 - \frac{3r^2}{4}x + \frac{sr^2}{4} = 0 \ldots\ldots\ldots\ldots [2]$$

Hence, c being given, find in the Tables the arc $3A$ corresponding to it, and, from the same Tables, take out the cosine of A; this latter value is the root of the equation [1]. If s be given, find $3A$ corresponding to the sine s, and then take from the Tables the sine of A, which is the root of the equation [2].

But, cubic equations have 3 roots; now, by Table, page 9, or by Cor. 6, page 28, if c = cos. $3A$, then also c = cos. $(2\pi - 3A)$ = cos. $(2\pi + 3A)$ = cos. $(4\pi - 3A)$, &c. hence, substituting instead of the arc A, the arcs $\frac{2\pi}{3} - A$, $\frac{2\pi}{3} + A$, $\frac{4\pi}{3} - A$, &c. any and all, of the following equations, are true.

$$cr^2 = 4\left(\cos.\ \frac{2\pi}{3} - A\right)^3 - 3 \cdot r^2 \cos.\ \left(\frac{2\pi}{3} - A\right) \ldots\ldots\ldots [a]$$

$$cr^2 = 4.\left(\cos.\ \frac{2\pi}{3} + A\right)^3 - 3 \cdot r^2 \cdot \cos.\ \left(\frac{2\pi}{3} + A\right) \ldots\ldots\ldots [b]$$

$$cr^2 = 4.\left(\cos.\ \frac{4\pi}{3} - A\right)^3 - 3 \cdot r^2 \cdot \cos.\ \left(\frac{4\pi}{3} - A\right) \ldots\ldots\ldots [d]$$

&c.

But, cos $\left(\frac{4\pi}{3} - A\right)$ = cos. $\left\{2\pi - \left(\frac{2\pi}{3} + A\right)\right\}$ =

$\cos. 2\pi . \cos. \left(\frac{2\pi}{3} + A\right) = \cos. \left(\frac{2\pi}{3} + A\right)$ [since $\sin. 2\pi = 0$, and $\cos. 2\pi = 1$]. Hence, the equation $[d]$ is precisely the same as the equation $[b]$.

Again, if we take the equation that would follow equation $[d]$, the cosine in it would be $\cos. \left(\frac{4\pi}{3} + A\right)$; now, $\cos. \left(\frac{4\pi}{3} + A\right) = \cos. \left\{2\pi - \left(\frac{2\pi}{3} - A\right)\right\} = \cos. 2\pi . \cos. \left(\frac{2\pi}{3} - A\right) = \cos. \left(\frac{2\pi}{3} - A\right)$: hence, this equation is precisely the same as equation $[a]$; and, in like manner, succeeding equations would recur, so that, essentially, there are only 3 different equations [1], $[a]$, $[b]$, and hence, in the equation

$$cr^2 = 4x^3 - 3r^2 x, \text{ or, } x^3 - \frac{3r^2}{4} x - \frac{cr^2}{4} = 0,$$

x may, either, $= \cos. A$, or, $= \cos. \left(\frac{2\pi}{3} - A\right)$, or, $= \cos. \left(\frac{2\pi}{3} + A\right)$, c being the cosine of $3A$.

The same reasoning applies to the equation [2], and similar conclusions will follow.

Example.

Let $x^3 - 147x - 285.5 = 0$; compare this with

$$x^3 - \frac{3r^2}{4} x - \frac{cr^2}{4} = 0;$$

$\therefore \frac{3r^2}{4} = 147$, and $r = 14$: and $c = \frac{285.5}{49}$ and the tabular cosine $= 10000 \frac{285.5}{49 \times 14}$

$= 4161.808$, &c. Now the arc corresponding to this cosine, or $3A$, is $65^\circ\ 24'$;

$\therefore A = 21° 48$ nat. cos. $= .9284858$ $\therefore$ to rad. $14 = 12.9988012$, 1st root.

Again,

$120° - 21° . 48' = 98°, 12'$, nat. cos $= - .1426289$;

$\therefore$ to rad. $14 = - 1.9968046$, 2d root.

Again,

$120° + 21° 48' = 141° . 48'$, nat. cos. $= - .7858569$;

$\therefore$ to rad. $14 = - 11.0019966$, 3d root,

and the sum of these two last roots, that is, of the 2d and 3d roots, exactly equals the first root, which ought to be the case, since the coefficient of the 2d term in the proposed equation is $= 0$.

But, cubic equations of any form may be solved by substituting Trigonometrical lines, sines, or tangents, for the quantities under the radical sign in Cardan's form: as Dr. Maskelyne has done, page 57, in his Introduction to Taylor's *Logarithms*.

Not only cubic, but quadratic equations may be solved by the aid of Trigonometrical Tables; and conveniently, when the coefficients are expressed by many figures.

Solution of a Quadratic Equation.

Let $x^2 + px - q = 0$,

then $x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)} = -\frac{p}{2}\left\{1 \mp \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}$

Assume $\frac{4q}{p^2} = \tan.^2 \theta$, then, $x = -\frac{p}{2}(1 \mp \sec. \theta)$.

Now by Prob. 3. Cor. 5, pp. 27, 28.

$$\cos. \theta = \frac{1 - \tan.^2 \frac{\theta}{2}}{1 + \tan.^2 \frac{\theta}{2}};$$

$$\therefore \sec. \theta = \frac{1 + \tan.^2 \frac{\theta}{2}}{1 - \tan.^2 \frac{\theta}{2}}$$

$$1 - \sec. \theta = -\frac{2 \tan.^2 \frac{\theta}{2}}{1 - \tan.^2 \frac{\theta}{2}}, \quad 1 + \sec. \theta = \frac{2}{1 - \tan.^2 \frac{\theta}{2}}$$

But, by Prob. 8, p. 55, l. 6,

$$\text{tan. } \theta = \frac{2 \text{ tan. } \frac{\theta}{2}}{1 - \text{tan.}^2 \frac{\theta}{2}}; \quad \therefore 1 - \text{sec. } \theta = -\text{tan. } \theta \,.\, \text{tan. } \frac{\theta}{2},$$

$$\text{and } 1 + \text{sec. } \theta = \frac{\text{tan. } \theta}{\text{tan. } \frac{\theta}{2}} = \text{tan. } \theta \,.\, \text{cot. } \frac{\theta}{2}.$$

Hence, one root $(\alpha) = \frac{p}{2} \text{ tan. } \theta \,.\, \text{tan. } \frac{\theta}{2}$,

the other $(\beta) = -\frac{p}{2} \text{ tan. } \theta \,.\, \text{cot. } \frac{\theta}{2}$.

And the whole formula of solution, in logarithms, is

$$\log. \text{ tan. } \theta = \tfrac{1}{2}(20 + \log. 4 + \log. q - 2 \log. p),$$

$$\text{and } \log. \alpha = \log. \frac{p}{2} + \log. \text{ tan. } \theta + \log. \text{ tan. } \frac{\theta}{2} - 20,$$

$$\text{and for 2d root, } \log. \beta = \log. \frac{p}{2} + \log. \text{ tan. } \theta + \log. \text{ cot. } \frac{\theta}{2} - 20.$$

The solution may be exhibited under a different form, thus:

$$x = \frac{p}{2}\left\{1 \pm \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\} = -\frac{p}{2}\left(1 \mp \frac{1}{\cos. \theta}\right)$$

$$= -\sqrt{q}\left(\frac{\cos. \theta \mp 1}{\text{tan. } \theta \,.\, \cos. \theta}\right) = -\sqrt{q}\left(\frac{\cos. \theta \mp 1}{\sin. \theta}\right)$$

$$= \text{(with the lower sign)} -\sqrt{q} \times \text{cot. } \frac{\theta}{2}, \text{ and}$$

$$= \text{(with the upper sign)} \sqrt{q} \,.\, \text{tan. } \frac{\theta}{2}.$$

Example. Let $x^2 + 13.56x - 72.31 = 0$.

Computation of θ.

10	= 10
log. 2	= .3010300
$\frac{1}{2}$ log. 72.31	= .9295992
	11.2306292
log. 13.56	= 1.1322597
log. tan. θ	= 10.0983695

$\therefore \theta = 51° . 26' . 2''$,

and $\frac{\theta}{2} = 25 . 43 . 1$.

Computation of positive root.

log. 6.78 =	.8312297
log. tan. 51° 26′ 2″ =	10.0983695
log. tan. 25 43 1 =	9.6827151
	20.6123143
$\therefore$ log. x =	.6123143
and x, or α =	4.09557

Computation of negative root.

log. 6.78 =	.8312297
log. tan. 51°.26′.2″ =	10.0983695
log. cot. 25 . 43 . 1 =	10.3172849
	21.2468841
$\therefore$ log. $(-x)$ =	1.2468841
and $-x$, or $-\beta$ =	17.6556

Hence the two roots are 4.09557 and -17.6556, and the sum of these two roots is -13.56, the coefficient of the second term of the equation, as it ought to be.

The equation that has been solved is $x^2 + px - q = 0$; if it had been $x^2 - px - q = 0$, the 2 roots would have been

$$\frac{p}{2}\left\{1 + \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}, \frac{p}{2}\left\{1 - \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}$$

that is, the 2 roots of the former equation taken negatively.

If the equation to be solved be $x^2 - px + q = 0$, then $x = \frac{p}{2}\left\{1 \pm \sqrt{\left(1 - \frac{4q}{p^2}\right)}\right\}$; assume $\frac{4q}{p^2} = \sin.^2\theta$ and $x = \frac{p}{2}(1 \pm \cos.\theta) = p . \cos.^2\frac{\theta}{2}$, or $= p \sin.^2\frac{\theta}{2}$.

Hence the rule of computation, logarithmically expressed,

is $\log. \sin. \theta = \frac{1}{2} (20 + 2 \log. 2 + \log. q - 2 \log. p)$ and

$\log. x = \log. p + 2 \log. \cos. \frac{\theta}{2} - 20$, or $= \log. p + 2 \log. \sin. \frac{\theta}{2} - 20$.

If the equation be $x^2 + px + q = 0$, its roots are the negatives of the roots of the preceding equation.

The preceding solutions are, in fact, the same as those given by Dr. Maskelyne, page 56, Introduction to Taylor's *Logarithms*.

If, in mathematical researches, equations, like those that have been given of the second and third degree, presented themselves to be solved, their solution would be conveniently effected by the preceding methods, and by the aid of the Trigonometrical Tables; but, the truth is, in the application of Mathematics to Physics, the solution of equations is an operation that very rarely is requisite, and consequently the preceding application of Trigonometrical formulæ is to be considered as a matter of curiosity rather than of utility.

SPHERICAL TRIGONOMETRY.

CHAP. VIII.

Definitions.

1. A SPHERE is a solid terminated by a curve surface, of which all the points are equally distant from an interior point, called the center of the sphere.

The surface of a sphere may be conceived to be generated by the revolution of a semicircle round its diameter.

2. Every section of a sphere, made by a plane, (so it will be demonstrated) is a circle. A *great* circle is that, the plane of which passes through the center; a *small* circle, that, the plane of which does not pass through the center.

3. The pole of a circle of a sphere, is a point in the surface, equally distant from every point of the circumference of the circle.

4. A plane is said to be a tangent of a sphere, when it has one point only common with the sphere.

5. A spherical triangle is a portion of the surface of a sphere included within three arcs* of three great circles, which arcs are called sides of the triangle.

---

* Each arc is supposed to be less than a semicircle, for the properties of a triangle that has its sides a, b, and $c = \pi + x$ are always known from those of a triangle that has its sides a, b, and a third side $= \pi - x$.

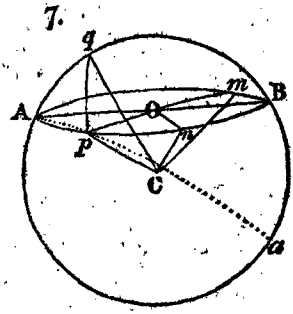
6. The angles of a spherical triangle, are the angles contained between the planes in which the arcs or sides lie. (See *Euclid*, Book XI. Def. 6.)

A spherical triangle is called rectangular, isosceles, equilateral, in the same cases that a plane triangle is.

Proposition I.

Every section of a sphere made by a plane is a circle.

Let $A\,n\,B\,m$ be the plane, draw CO * perpendicular to it, which consequently, (by *Euclid*, Book XI. Def. 3.) is perpen-



dicular to every straight line meeting it in that plane; hence, since

$$Cn = Cm, \text{ and, } \angle Con = \angle Com = 90°.$$
$$On^2 = Cn^2 - CO^2,\ Om^2 = Cm^2 - CO^2;\ \therefore On = Om,$$

and similarly, $On = Op$; $\therefore Om, On, Op$ are equal; $\therefore ApBm$ is a circle, and O is its center.

Cor. 1. When $CO = 0$, or when the plane passes through the

* CO is not drawn in the diagram.

center C of the sphere, $On = Om = Cm = CB$, the radius of the sphere.

Cor. 2. Hence two great circles always bisect each other, for their common intersection passing through the center is a diameter.

Cor. 3. Through two points on the surface of a sphere, such as A, p, a great circle, (part of which is represented in the Figure by the dotted curve from A to p) may be made to pass; for, the two points A, p, with C the center as a third point, determine the position of a plane, the intersection of which with the surface of a sphere is a great circle passing through the points A, p. In like manner a great circle may pass through the points p, q.

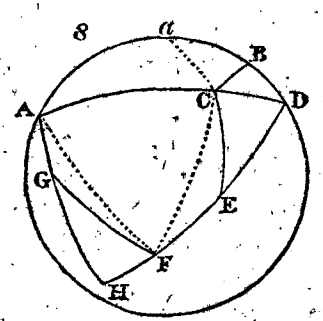
If the two points, insteadof being A, p, be A, a, the two ends of a diameter, then the three points A, C, a, lying in the same line, do not determine the position of a plane; but, through these three points, innumerable planes may pass.

Prop. II. In every spherical triangle as Apq, (Fig. p. 116.) any one side is less than the sum of the two others.

For the arcs Ap, Aq, pq, measure the angles ACp, ACq, pCq; but by *Euclid*, Book XI. Prop. 20, any angle, as ACp *, is less than the two others ACq, pCq; $\therefore Ap$ is less than $Aq + pq$.

Prop. III. The sum of the three sides of a spherical triangle is less than the circumference of a great circle (2π).

Let ACB be the spherical triangle; then, $CB < CD + BD$



by the former Proposition, and $AC + AB = AC + AB$; $\therefore$

* Conceive, in Fig. p. 116, a straight line to be drawn from A to the center C.

$AC + CB + AB < (AC + CD) + (BD + AB)$; $\therefore$ $< ACD + ABD$; $\therefore$ $< (\pi + \pi)$; $\therefore$ $< 2\pi$.

Cor. Hence, the sum of the sides of a polygon $ACEFGA$ (the sides being arcs of great circles) is less than 2π.

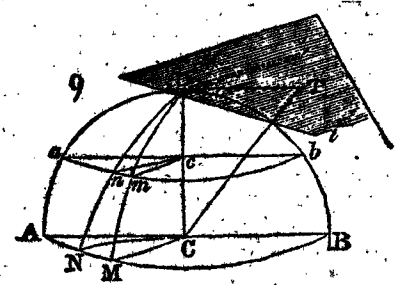
For, by the Proposition, $AD + DH + AH$ is $< 2\pi$;

or, $AC + (CD + DE) + EF + (FH + HG) + AG < 2\pi$,

but $CD + DE > CE$, and, $FH + HG > FG$;

$\therefore$ *a fortiori*, $AC + CE + EF + FG + AG < 2\pi$.

Prop. IV. If CD be drawn from the center C, perpendicularly to the plane of the circle $ANMB$, then, D is the pole of the



circle $ANMB$ and of all small circles, such as $anmb$, that are parallel to it.

For, since DC is perpendicular to the plane of $ANMB$, it is perpendicular to all lines in it, as CN, CM, &c. (See *Euclid*, Book XI. Def. 3.) Hence, DCA, DCN, DCM, each $= 90°$; consequently, since DC is common, and CA, CN, CM, are equal, the hypothenuses, which are the chords of the arcs DA, DN, DM, are equal; $\therefore$ the arcs DA, DN, DM are equal to one another and to 90°, and $\therefore$ D, (by Def. 3.) is the pole of $ANMB$.

Again, in the small circle $anmb$, ca cn, cm, are equal, as are the angles Dca, Dcn, Dcm; $\therefore$ as before the chords of Da, Dn, Dm, are equal, and the arcs Da, Dn, Dm; $\therefore$ D is equally distant from every point in the circumference of $anmb$, and therefore is its pole.

Cor. 1. By Definition 6, the spherical angle AMD is equal to the inclination of the planes $AMBC$, DCM, and therefore is a right angle.

Cor. 2. Hence, to find the pole of a great circle, draw, from the point M, a great circle perpendicular to AM, and take $MD = 90°$; then D is the pole: and, reversely, if D be the pole, DMN, DNM are right angles, and DN, DM are quadrants.

Cor. 3. Hence, to describe a great circle, of which D is the pole, take DN, DM, each $= 90°$; then let a plane pass through N, M, and C, and its intersection with the surface of the sphere is the circle required.

Cor. 4. Since NDM may be of any magnitude from 0 to 180°, and since the angles DNM, DMN, are, each $= 90°$, the sum of the three angles of a triangle, as DNM, may be any angle between 180° and 360°, which are the two limits.

Cor. 5. The radius of the small circle $anmb$ is Cn, which is the sin. Dn, or, cos. Nn; and, if the great circle $ANMB$ be divided into any number of equal parts, each equal to NM, the small circle $anmb$ will be divided into the same number of equal parts, each part being equal to nm; but, the magnitude of nm will be to the magnitude of NM, as the circumference of $anmb$ to that of $ANMB$, consequently, as the radius Cn to the radius CN; or, as sin. Dn to sin. DN; as sin. Dn to radius; or, as cos. Nn to radius.

Prop. V. If a plane TDt is perpendicular to CD, it is a tangent to the sphere.

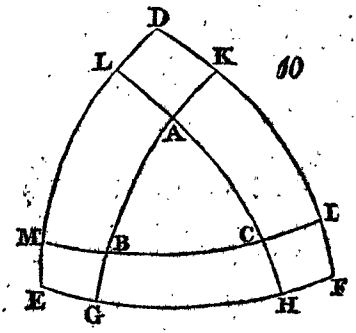
For take any point T, join DT, then CDT is a right angle; $\therefore$ CT is greater than CD; $\therefore$ T is without the surface, and since this is true of every point in the plane, except the point D, the plane TDt (by Def. 4.) is a tangent to the sphere.

Prop. VI. If DT, Dt be drawn, in the planes DCA, DCN respectively, tangents to the arcs Da, Dn, at the point D, the angle TDt is equal to the angle ADN, both of which angles are measured by the arc AN.

For, DT, Dt are perpendicular to DC, which is the intersection of the planes DCA, DCN; $\therefore$ by *Euclid*, Book XI. Def. 6. TDt measures the inclination of the planes, and therefore is equal to the angle ADN; but TDt, the inclination of the planes, $= ACN$, of which AN is the measure; $\therefore$ AN is the measure of TDt and of ADN.

Cor. If two arcs of great circles intersect each other, their vertical angles are equal.

Prop. VII. If from the points A, B, C, of the triangle ABC, as poles, be described, the arcs EF, FD, DE, forming a triangle



DEF; then, reciprocally, the points D, E, F are poles of the arcs BC, AC, AB.

Since A is the pole of EF, the arc of a great circle from A, to any point in EF, and therefore to a point as E, is a quadrant; similarly, since C is the pole of DE, the arc of a great circle from C, to any point in DE, and therefore to E, is a quadrant. Hence, E is distant from two points A, C, in an arc AC, by the quantity of a quadrant; $\therefore$ by Cor. 2, Prop. 4, E is the pole of AC; and similarly, F is the pole of AB, and D of BC.

Prop. VIII. The former construction remaining, the measures of the angles at A, B, C, are the supplements of the sides opposite, that is, of EF, DF, DE; and, reciprocally, the measures of the angles at D, E, F, are the supplements of the sides BC, AC, AB.

For, the measure of the angle at A (see Prop. 6.) is GH. Now, $GH = EH - EG = EH - (FE - FG) = EH + FG - FE = 90° + 90° - FE = 180° - FE =$ (by Art. 5, page 4.) the supplement of FE.

Again,

the measure of the angle at B, or $KI = FK - FI = FK - (DF - DI) = FK + DI - DF = 90° + 90° - DF = 180° - DF =$ supp^t. DF; and similarly, LM, the measure of the angle at C, is the supplement of DE.

Secondly, the measure of the angle at D, or $MI = MC + CI = MC + IB - BC = 90° + 90° - CB = 180° - CB =$ supp. CB; and similarly, LH, KG, the measures of the angles at E and F are the supplements of AC and AB.

From its properties, the triangle DEF has been called, by English Geometers, the *Supplemental Triangle*; and from the mode of its description, by French, the *Polar Triangle*.

Cor. 1. If any angle as B, $= 90°$, $DF = 180° - 90° = 90°$, or is a quadrant: and if B and C each $= 90°$, DF, DE, each is equal to a quadrant.

Cor. 2. If (Cor. 4. Prop. 4), the angles at B and C each $= 90°$, and the angle at A is nearly $180° = 180° - \varkappa$, $\varkappa$ being a very small angle, then the side of the supplemental triangle opposite to A is equal to $\varkappa$; and the sum of the three sides of the supplemental triangle = semicircle $+ \varkappa$, $= 180° + \varkappa$.

Prop. IX. The sum of the three angles of a spherical triangle is $>$ 2 right angles $<$ 6 right angles.

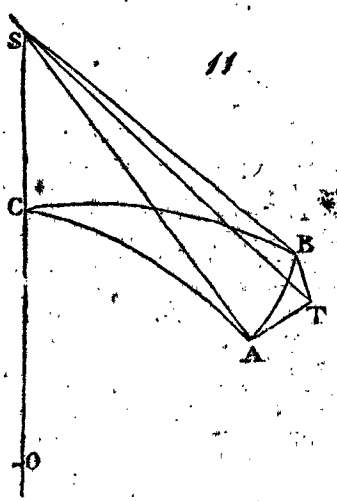
For, the angles of the triangle + sides of the supplemental triangle $= 180° + 180° + 180° = 6 \times 90°$; $\therefore$ since the sides of the supplemental triangle must have some magnitude, the angles of the triangle must be less than $6 \times 90°$: again, by Prop. 3, p. 117, the sides of any triangle, and $\therefore$ the sides of the supplemental triangle, are less than $4 \times 90°$, $= 4 \times 90 - \varkappa$, suppose; consequently, the angles of the triangle $= 6 \times 90° - (4.90° - \varkappa) = 2 \times 90° + \varkappa$.

Cor. 1. Hence a spherical triangle may have two or three right angles, two or three obtuse angles.

Cor. 2. If the angles at B and C are right angles, AB, AC are quadrants, and A is the pole of BC: if also the angle at A is a right angle, the triangle ABC coincides with the supplemental or polar triangle, and the triangle ABC is contained eight times in the surface of the sphere.

Prop. X. The angles at the base of an isosceles spherical triangle are equal.

In the triangle ACB, let $AC = BC$, draw the tangents AS, BS, which are equal and which cut their secant OS in the common



point S. Draw also from A and B two tangents AT, BT, which, by *Euclid*, Book III. Prop. 37. are equal.

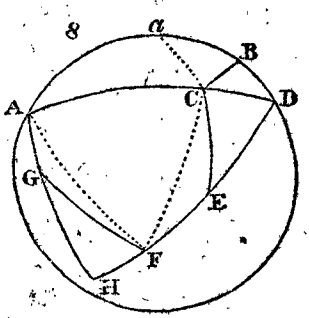
Hence, in the triangles SAT, SBT; SA, AT, ST, are respectively equal to SB, BT, ST; $\therefore$ by *Euclid*, Book I, Prop 8, the angle SAT = the angle SBT, and $\therefore$ by Prop. 6, page 120, the spherical angle at A = the spherical angle at B.

To prove the reverse proposition, that is: if the angles are equal, the sides are equal; take the supplemental triangle, then,

since its sides opposite to the angles at A and B are supplemental of equal angles, the two sides are equal, and the supplemental triangle is isosceles; $\therefore$ by this proposition, the angles at the base are equal; $\therefore$ their supplements which are the sides AC, BC, are also equal.

PROP. XI. In a spherical triangle the greater side is opposite to the greater angle.

In the triangle ACB, let $\angle ACB$ be greater than ABC;



make $BCa = CBa$; $\therefore Ca = aB$ (Prop. 10.); but $Aa + aC$ is greater than AC; $\therefore Aa + aB$, or AB is greater than AC, by Prop. 2, page 117.

PROP. XII. The surface of the sphere included between the arcs DN, DM, is proportional to the angle NDM or the arc NM. See Fig. 9.

For, if the circumference $ANMB$ be divided into equal parts as NM, and great circles be drawn from D through the points of division as N, M, the portions of the surface, such as NDM, will be all similar and equal; hence, if AM contains NM, p times, or, if $AM = p \times NM$, the surface ADM will $= p \times NDM$.

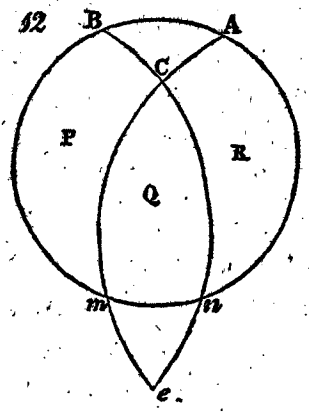
COR. When DM coincides with DB, the angle ADB and its measure $AMB = 180°$: hence, if $S =$ the whole surface of a sphere, and if $a =$ the angle NDM, or the arc NM, the surface

$NDM = \frac{S}{4} \times \frac{a}{180}$; or, since S = area of 4 great circles of a sphere, (See Simpson's *Fluxions*, page 189, vol. I, or Vince's, page 89), $= 4\pi$ ($\pi = 3.14159$, &c.) the surface $NDM = \pi \frac{a}{180}$.

The surface may be differently expressed, thus: π is the circumference of a circle the radius of which is $\frac{1}{2}$; $\therefore 2\pi$ represents the circumference of a circle, such as $ANMB$, the radius of which is supposed $= 1$, and as we represent arcs of great circles by the number of degrees which they contain, $2\pi = 360°$, and $\frac{S}{4} = \pi =$ $180°$: hence, the surface $NDM = 180° \cdot \frac{a}{180} = a°$.

PROP. XIII. The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.

Let the triangle be ABC, a, b, c, representing the magnitudes of the angles at A, B, C; let P = surface $BC\,m\,B$, $Q = m\,C\,n\,m$, $R = AC\,n\,A$; produce the arcs $C\,m$, $C\,n$ till they meet at e



(which will be on the hemisphere opposite to that represented by $AB\,m\,n\,A$) then, each of the angles at C and e, equals the angle of the planes in which the arcs $C\,m\,e$, $C\,n\,e$, lie; $\therefore$ the angles at C and e are equal.

Again, the semicircles ACm, Cme; BCn, Cne are equal; or, $AC + Cm = Cm + me$, and $\therefore AC = me$, and $BC = ne$: and the triangle men = the triangle ABC; let x = its area, then, by Cor. to Prop. 12.,

$$x' + P = \frac{S}{2} \cdot \frac{a}{180}$$

$$x + Q = \frac{S}{2} \cdot \frac{c}{180}, \text{ and, } x + P + Q + R = \frac{S}{2}$$

$$x + R = \frac{S}{2} \cdot \frac{b}{180};$$

consequently, by addition,

$$2x + (x + P + Q + R) \text{ or } 2x + \frac{S}{2} = \frac{S}{2} \cdot \left(\frac{a + b + c}{180}\right)$$

$$\therefore x = \frac{S}{4 \cdot (180)}(a + b + c) - \frac{S}{4} = a + b + c - 180°,$$

by the Cor. to the last Proposition.

This result is the same as that given by Caswell, in Wallis's Works, vol. II. page 875, who attributes the Theorem to Albert Girard *.

Cor. 1. Since by Prob. 9, page 121, the limits of $a + b + c$ are 180° and 540°, the area of the triangle ABC may be equal to any number of degrees between 0 and 360°.

Cor. 2. If each of the angles a, b, c, = 90°, the area of the triangle ABC = 270° − 180° = 90°, which is $\frac{1}{8}$th of the whole surface, since $4\pi = 8\,(90°)$: this agrees with Cor. 2, Prop. 9.

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* This expression for the value of the area was a merely speculative truth, and continued barren for more than 150 years, till 1787, when General Roy employed it, in correcting the spherical angles of observation made in the great Trigonometrical Survey, *Phil. Trans.* vol. VIII, year 1790, page 163. See also *Mem. Acad.* Paris, 1787, page 358, and *Mem. Inst.* vol. VI, p. 511.

Cor. 3. The area of DNM (see Fig. 9), $= \frac{\delta}{4} \times \frac{e}{180}$ (if e be its angle), $= e$; if we conceive DN, DM, to be continued till they meet in the opposite pole, and name the space included between the great circles *a Lune*, then the lune $= 2 \cdot DNM = 2e$: but the area of $ABC = a + b + c - 180°$; equate this with the area of the lune, and

$$e = \frac{a + b + c}{2} - 90°;$$

which is the value of the angle NDM, when the area of the lune equals the area of the triangle.

Prop. XIV. If n be the number of the sides of a spherical polygon, its surface equals the sum of its angles, minus the product of two right angles multiplied by $n - 2$.

Let the polygon be $AGFECA$: divide it by the means of the arcs of great circles into triangles, then

$$\text{area } AGF = \angle AGF + \angle GFA + \angle GAF - 180°$$
$$\text{area } AFC = \angle FAC + \angle AFC + \angle ACF - 180°$$
$$\text{area } CFE = \angle CEF + \angle CFE + \angle FCE - 180°$$

$\therefore$ area of polygon $= \angle AGF + (\angle FAC + \angle GAF) + (\angle GFA + \angle AFC + \angle CFE) + \angle CEF + (\angle FCE + \angle ACF) - 3 \times 180°$

$= \angle AGF + \angle GAC + \angle GFE + \angle CEF + \angle ECA - (5 - 2)\ 180°.$

This demonstration proves the Proposition to be true for a polygon of five sides, and a similar demonstration will prove it true for a polygon of n sides: for, it is plain, if, instead of AC, we introduce Ca, Aa, that is, if we introduce an additional side, we introduce an additional triangle, and consequently we must introduce an additional 180° to be subtracted, that is, the negative part will become $- \{(5 - 2)\ 180° + 180°\}$ or $- (6 - 2)\ 180°$.

The preceding Propositions belong, more properly, to Spherical Geometry than to Spherical Trigonometry; they have, however, been here inserted, because they exhibit certain properties

of spherical triangles rather curious, and very easy of demonstration; so that, if not of essential use in assigning the relations between the angles and sides of spherical triangles, which it is the special object of Trigonometry to assign, they will not materially divert or impair the Student's attention.

In the next Chapter, we will proceed to deduce those formulæ by which, the relations between the sides and the angles of spherical triangles are expressed.

CHAP. IX.

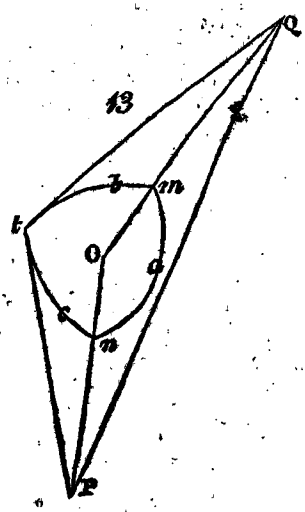
On the Expressions for the Cosine and Sine of the Angle of a Spherical Triangle in Terms of the Sines and Cosines of the Sides.

We will begin this Chapter, with establishing in the following Problem, the fundamental formula, from which all the methods and forms of solution will be deduced: it corresponds to the fundamental one of Plane Trigonometry, inserted in page 20; and the Student, who understands, in principle, the use made of that latter formula, possesses, in fact, the clue to the subsequent demonstrations of Spherical Trigonometry.

PROP. XV. PROBLEM.

It is required to express the cosine of the angle of a spherical triangle in terms of the sines and cosines of the sides.

Let the triangle be $m\,t\,n$, let the sides be a, b, c, and the



opposite angles A, B, C; conceive O to be the center of the sphere,

and draw the two tangents tQ, tP, at the point t, to the arcs tm, tn; then $\angle QtP = \angle A$: by Prop. 6; and the angle at O is measured by mn, or a; and, by a definition of the secant,

$$OQ \text{ is sec. } b$$
$$OP \text{ is sec. } c$$

The principle of the demonstration that follows, is, to obtain two values of QP, one from the triangle POQ, the other from PtQ, and then to compare them: now by *Euclid*, Prop. 13, Book II, or Prob. 1 of this Treatise:

in $\triangle POQ$, $PQ^2 = \text{sec.}^2 b + \text{sec.}^2 c - 2 \text{ sec. } b \text{ . sec. } c \cos. a$ [rad. 1]
in $\triangle PtQ$, $PQ^2 = \text{tan.}^2 b + \text{tan.}^2 c - 2 \text{ tan. } b \text{ . tan. } c \text{ . cos. } A$.

Subtract the lower expression from the upper,

then, since $\text{sec.}^2 b - \text{tan.}^2 b = (\text{rad.})^2 = 1$, we have

$0 = 1 + 1 + 2 \text{ tan. } b \text{ . tan. } c \text{ . cos. } A - 2 \text{ sec. } b \text{ . sec. } c \text{ . cos. } a$

and hence,

$$\cos. A = \frac{\cos. a \text{ . sec. } b \text{ . sec. } c - 1}{\text{tan. } b \text{ . tan. } c} = \frac{\cos a - \cos. b \text{ . cos. } c}{\cos. b \cos. c \text{ . tan. } b \text{ . tan. } c}$$

$$= [\text{since cos. } b \text{ . tan. } b = \text{sin. } b] \frac{\cos. a - \cos. b \text{ . cos. } c}{\text{sin. } b \text{ . sin. } c} \ldots\ldots\ldots [a]$$

and similarly, since the process for finding cos. B, cos. C, will be exactly the same, changing a for b, b for a, &c. the result must be similar, that is,

$$\cos. B = \frac{\cos. b - \cos. a \text{ . cos } c}{\text{sin. } a \text{ . sin. } c} \ldots\ldots\ldots [b]$$

$$\cos. C = \frac{\cos. c - \cos. a \text{ . cos. } b}{\text{sin. } a \text{ . sin. } b} \ldots\ldots\ldots [c]$$

Cor. 1. Hence, since

$$\cos. c = \text{sin. } a \text{ . sin. } b \text{ . cos. } C + \cos. a \text{ . cos. } b$$

by substituting this value of cos. c in the expression $[a]$, we have

$$\cos. A = \frac{\cos. a - \cos. b \text{ . sin. } b \text{ . sin. } a \text{ . cos. } C - \cos. a \cos.^2 b}{\text{sin. } b \text{ . sin. } c}.$$

But, $\cos. a - \cos. a \text{ . cos.}^2 b = \cos. a (1 - \cos.^2 b) = \cos. a \text{ . sin.}^2 b$;

$$\therefore \cos. A = \frac{\cos. a \sin.^2 b - \cos. b . \sin. b . \cos. C . \sin. a}{\sin. b . \sin. c}$$

$$= \frac{\cos. a . \sin. b - \cos. b . \sin. a . \cos. C}{\sin. c};$$

similarly, $\cos. B = \frac{\cos. b . \sin. a - \cos. a . \sin. b . \cos. C}{\sin. c}$;

$$\therefore \cos. A + \cos. B = \frac{(\cos. a . \sin. b + \cos. b . \sin. a)(1 - \cos. C)}{\sin. c}$$

$$= \frac{\sin. (a+b) 2 \sin.^2 \frac{C}{2}}{\sin. c}.$$

Prop. XVI. Problem.

It is required to express the sine of the angle of a spherical triangle in terms dependent on its sides.

By the last Proposition, $\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}$;

$$\therefore 1 + \cos. A =$$

$$\frac{\cos. a - (\cos. b . \cos. c - \sin. b . \sin. c)}{\sin. b . \sin. c} = \frac{\cos. a - \cos. (b+c)}{\sin. b . \sin. c}$$

by form [2], p. 25; but by form [8], page 27,

$$\cos. a - \cos. (b + c) = 2 \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{b + c - a}{2}\right).$$

Hence,

$$1 + \cos. A = \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a + b + c}{2}\right) \sin. \left(\frac{b + c - a}{2}\right)$$

$$\text{or, } = \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a+b+c}{2}\right) . \sin. \left(\frac{a+b+c}{2} - a\right)$$

Similarly,

$$1 - \cos. A = \frac{\cos. b . \cos. c + \sin. b \sin. c - \cos. a}{\sin. b . \sin. c}$$

$$= \frac{\cos.(b-c) - \cos. a}{\sin. b \,.\, \sin. c} \text{ (by form [4], p. 26,)}$$

$$= \frac{2}{\sin. b \sin. c} \times \sin.\left(\frac{a+b-c}{2}\right) \sin.\left(\frac{a+c-b}{2}\right) \text{ by form [8] p. 27;}$$

$$\text{or, } = \frac{2}{\sin. b \,.\, \sin. c} \times \sin.\left(\frac{a+b+c}{2} - b\right) \sin.\left(\frac{a+b+c}{2} - c\right).$$

Hence, if we multiply together $1 + \cos. A$, and $1 - \cos. A$, the product of which is $\sin.^2 A$, and substitute S instead of $\frac{a+b+c}{2}$, we shall have $\sin.^2 A = \frac{4}{\sin.^2 b \,.\, \sin.^2 c} \times \sin. S \,.\, \sin.(S-a) \sin.(S-b) \sin.(S-c)$, and consequently,

$$\sin. A = \frac{2}{\sin. b \,.\, \sin. c} \sqrt{\{\sin. S \,.\, \sin.(S-a) \sin.(S-b) \sin.(S-c)\}} \,*$$

Cor. 1. If we wish to compute $\sin. B$, we must begin from $\cos. B = \frac{\cos. b - \cos. a \,.\, \cos. c}{\sin. a \,.\, \sin. c}$, and proceed exactly in the steps of the former process: the result will be a fraction, the numerator of which is the numerator of the above fraction for the $\sin. A$, and the denominator will be $\sin. a \sin. c$; call the common numerator N, then

$$\sin. A = \frac{N}{\sin. b \,.\, \sin. c}, \quad \sin. B = \frac{N}{\sin. a \,.\, \sin. c}, \quad \sin. C = \frac{N}{\sin. a \,.\, \sin. b}$$

Cor. 2. Hence $\frac{\sin. A}{\sin. B} = \frac{N}{\sin. b \,.\, \sin. c} \times \frac{\sin. a \,.\, \sin. c}{N} = \frac{\sin. a}{\sin. b}$, and $\frac{\sin. A}{\sin. C} = \frac{\sin. a}{\sin. c}$;

or, if these equations be expressed, after the manner of expres-

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* The Student, for his own convenience, is desired to compare these expressions and the manner of deducing them, with the corresponding ones in Plane Trigonometry.

sing a proposition, *the sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

Right-angled spherical triangles may be considered as particular cases of oblique. The solutions of the latter, then, would necessarily include those of the former; and, accordingly, if we wished to generalise as much as we might generalise, it would not be requisite separately to consider them. Since, however, it is our object to render investigation as simple and as easy as it is possible to the Student, we shall not avail ourselves of this abridgment, but proceed, in the ensuing Chapter, to treat distinctly of the cases of right-angled spherical triangles.

CHAP. X.

Formulæ of Solution for Right-angled Spherical Triangles.—Affections of Sides and Angles.—Circular Parts.—Naper's Rules.—Quadrantal Triangles.—Examples.

PROP. XVII. PROBLEM.

IT is required to investigate formulæ of solution for all the cases of right-angled spherical triangles *.

These are to be derived from the fundamental expressions [a], [b], [c], given, page 129: let C be the right angle and $= 90°$; $\therefore$ cos. $C = 0$;

hence,

$$\text{cos. } C, \text{ or } 0 = \frac{\text{cos. } c - \text{cos. } a \,.\, \text{cos. } b}{\text{sin. } a \,.\, \text{sin. } b}, \text{ and } \therefore \text{ cos. } c = \text{cos. } a \,.\, \text{cos. } b \; [1]$$

hence, $\text{cos. } a = \frac{\text{cos. } c}{\text{cos. } b}$, $\text{cos. } b = \frac{\text{cos. } c}{\text{cos. } a}$; substitute these values respectively, for cos. a and cos. b, in the expressions for cos. A, cos. B, and we have

$$\text{cos. } A = \frac{\text{cos. } a - \text{cos. } b \,.\, \text{cos. } c}{\text{sin. } b \,.\, \text{sin. } c} = \frac{1}{\text{sin. } b \,.\, \text{sin. } c} \left(\frac{\text{cos. } c}{\text{cos. } b} - \text{cos. } b \,.\, \text{cos. } c \right) =$$

$$\frac{\text{cos. } c\,(1 - \text{cos.}^2 b)}{\text{cos. } b \,.\, \text{sin. } b \,.\, \text{sin. } c} = \frac{\text{cos. } c \,.\, \text{sin.}^2 b}{\text{cos. } b \,.\, \text{sin. } b \,.\, \text{sin. } c} =$$

* Spherical triangles that have one right angle only, are the subject of investigation: and spherical triangles that have two right angles, and three right angles, are excluded.

$$\frac{\cos.\ c\ .\ \sin.\ b}{\cos.\ b\ .\ \sin.\ c} = \cot.\ c\ .\ \tan.\ b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

similarly,

$$\cos.\ B = \frac{\cos.\ c(1-\cos.^2 a)}{\cos.\ a\ .\sin.\ a\ .\ \sin.\ c} = \frac{\cos.\ c\ .\ \sin.\ a}{\cos.\ a\ .\ \sin.\ c} = \cot.\ c\ .\ \tan.\ a \ldots [3]$$

Now in [1] a, b, c are involved, and two of these being given the third may be found: in [2] A, c, b are involved: and if we choose, symbolically, to represent by (A, c, b) the form in which three quantities as A, c, b, are involved, then, similarly, the other several forms that can arise by combining angles and sides may be thus represented:

(c, a, b); $\{(A, c, b), (B, c, a)\}$ $\{(B, c, b)\ (A, c, a)\}$

$\{(A, a, b)\ (B, a, b)\}$ $\{(A, B. a)\ (A, B, b)\}$; (A, B, c):

which are in number 10, as they ought to be; the combination in 5 things, 3 and 3 together, being $\frac{5.4.3}{1.2.3} = 10$: those combinations that are similar, such as the second and third, are included within brackets.

The forms [1], [2], [3], have been already deduced, and the remaining ones must be deduced from them, and from the form $\frac{\sin.\ A}{\sin.\ B} = \frac{\sin.\ a}{\sin.\ b}$, by the common process of substitution and elimination,

$\left\{\begin{matrix}(c, a, b), \\ (A, c, b) \\ (B, c, a)\end{matrix}\right.$ that is, $\begin{matrix}\cos.\ c = \cos.\ a.\ \cos.\ b \ldots\ldots [1] \\ \cos.\ A = \cot.\ c.\ \tan.\ b \ldots\ldots [2] \\ \cos.\ B = \cot.\ c.\ \tan.\ a \ldots\ldots [3]\end{matrix}$

(B, c, b)

this is Cor. to Prop. 16, for $\frac{\sin.\ B}{\sin.\ C} = \frac{\sin.\ b}{\sin.\ c}$; $\therefore$ since $\sin.\ C = 1$,

$$\sin.\ B = \frac{\sin.\ b}{\sin.\ c}, \text{ or } \sin.\ b = \sin.\ c.\ \sin.\ B \ [4]$$

(A, c, a) is exactly similar;

$$\therefore \sin.\ A = \frac{\sin.\ a}{\sin.\ c}, \text{ or } \sin.\ a = \sin.\ c.\ \sin.\ A \ [5]$$

(A, a, b): cos. A = cot. c. tan. b by [2], but cot. $c = \frac{\cos. c}{\sin. c}$,

and cos. c = cos. a . cos. b by [1], and sin. $c = \frac{\sin. a}{\sin. A}$ by [5] p. 134,

$\therefore$ cos. A = tan. b . $\frac{\cos. a . \cos. b . \sin. A}{\sin. a}$,

$\therefore$ cot. A = cot. a . sin. b.....[6]

(B, a, b) is similar, and $\therefore$ cot. B = cot. b . sin. a.....[7]

(A, B, a): divide [2] by [4], then $\frac{\cos. A}{\sin. B} = \frac{\cos. c}{\cos. b}$ = cos. a

from [1], or, cos. A = cos. a . sin. B.....[8]

(A, B, b) is similar, and cos. B = cos. b . sin. A.....[9]

(A, B, c): multiply together [6] and [7]

and cot. A. cot. B = cos. b. cos. a = cos. c.....[10]

Thus are all the forms easily deduced from [1], [2], [3]. Indeed, the quantities that are to be combined, indicate plainly the proper process of elimination; but, these forms, although very simple and fitted for logarithmic computation, are not easily remembered, and therefore an artificial memory has been supplied to the Student and Computist, by rules, known by the title of *Naper's Rules for Circular Parts*; and in the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that, which is the proper object of rules, namely, facility and brevity of computation.

The rules and their description are as follow:

Description of Circular Parts.

The right angle is not considered; the complements of the two other angles, the complement of the hypothenuse, the two sides, making in all five quantities, are called by Naper, *circular parts*. Any one of these circular parts may be called a *middle part* (M), and, then, the two circular parts immediately adjacent to the right and left of M are called *adjacent parts*; the other two remaining circular parts, each separated from M the middle part,

by an adjacent part, are called *opposite parts*, or *opposite extremes*: thus, let the side a be M,

then, comp. B, or $90° - B$, and b are adjacent parts,
and, comp. c, or $90° - c$, and $90° - A$, opposite parts.

If $90° - A$ be M,

$90° - c$ and b are adjacent parts; $90° - B$ and a, opposite parts.

This necessary explanation being premised, we come to

Naper's Rules.

1. The rectangle of the sin. M and radius = rectangle of the tangents of adjacent parts.

2. The rectangle of the sin. M and radius = rectangle of the the cosines of the opposite parts.

There is no separate and independent proof of these rules, but the rules will be manifestly just, if it can be shewn that they comprehend every one of the ten results, [1], [2], [3], &c. given, in pages 134, 135; for, those results solve every case in right-angled spherical triangles.

Proof of Naper's Rules.

		M.	Adjacent Parts.	Opposite Parts.
1.	cos. c = cos. a . cos. b, or, sin. $(90°-c)$ = cos. a . cos. b	$90°-c$		a, b
2.	cos. A = cot. c . tan. b, or, sin. $(90°-A)$ = tan. $(90°-c)$ tan. b	$90°-A$	$90°-c$, b	
3.	cos. B = cot. c . tan. a, or, sin. $(90°-B)$ = tan. $(90°-c)$ tan. a	$90°-B$	$90°-c$, a	
4.	sin. b = sin. c . sin. B, or, sin. b = cos. $(90°-c)$ cos. $(90°-B)$	b		$90°-c$, $90°-B$
5.	sin. a = sin. c . sin. A, or, sin. a = cos. $(90°-c)$ cos. $(90°-A)$	a		$90°-c$, $90°-A$
6.	cot. A = cot. a . sin. b, or, sin. b = tan. $(90°-A$. tan. a	b	$90°-A$, a	
7.	cot. B = cot. b . sin. a, or, sin. a = tan. $(90°-B)$. tan. b	a	$90°-B$, b	
8.	cos. A = cos. a . sin. B, or, sin. $(90°-A)$ = cos. a . cos. $(90°-B)$	$90°-A$		a, $90°-B$
9.	cos. B = cos. b . sin. A, or, sin. $(90°-B)$ = cos. b . cos. $(90°-A)$	$90°-B$		b, $90°-A$
10.	cot. A . cot. B = cos. c, or, sin. $(90°-c)$ = tan. $(90°-A)$ tan. $(90°-B)$	$90°-c$	$90°-A$, $90°-B$	

This is a complete proof of the truth of the rules, and, as we have already said, the only kind of proof which the rules admit of; but, after the proof, the rules ought to be used, and the formulæ having performed the service of proving them, are then superseded. The rules ought to be used also, not only in the immediate solution of right-angled spherical triangles, but in deducing, where they can be made subservient, the properties of oblique triangles. We here allude to those properties announced in the Propositions 24, 25, 26, 27 of Robert Simson's *Trigonometry*. These immediately appear on the application of Naper's Rules, and their deduction is so obvious, that it is, practically, against the interest of the Student to make them the subject of three or four formal Propositions; since it is not worth the while to burthen the memory with the terms and enunciation of a Proposition, for the sake of formally making one or two steps in the process of deduction.

The forms for the solution of right-angled triangles have been deduced from the general expressions for the cos. A, cos. B, cos. C, in which, as in a particular case, $C = 90°$. The same general expressions may also be used for any other case, in which, to any one or more of the sides or angles particular values should be assigned: for instance, the side c may be a quadrant $= 90°$, in which case the triangle has been called a quadrantal triangle; in this case

$$\cos. C = \frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b} = -\frac{\cos. a . \cos. b}{\sin. a . \sin. b} \text{ [since cos. } c = 0]$$

and $\therefore$ sin. $(90° - C) = -$ cot. a . cot. b

$$= -\tan. (90° - a) \tan. (90° - b) \ldots\ldots [a]$$

$$\text{Again, cos. } B = \frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c} = \frac{\cos. b}{\sin. a}$$

$$\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c} = \frac{\cos. a}{\sin. b}$$

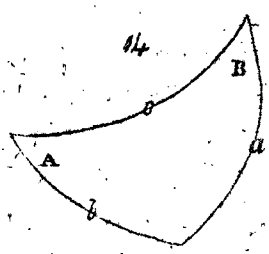
From these three equations, and the additional one, viz. $\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$, may be deduced all the cases in quadrantal, as they have been, in right-angled spherical triangles.

If, in the above expression [*a*] we call $90° - C$ the middle part, and $90° - a$, $90° - b$ which are the complements of the sides, the adjacent parts, the above result might be comprehended under a Rule like Naper's; but, if a case like that occurs, the surest method, and, perhaps, on the whole, the most expeditious for the Student and computist, would be to take the supplemental or polar triangle, and to solve it by Naper's Rule; for the angle in the supplemental triangle opposite to $c = 180° - c = 180° - 90°$, is a right angle: thus, in the case adduced, the angles of the supplemental triangle are $180° - a$, $180° - b$, and the hypothenuse is $180° - C$; therefore by Naper's first rule, $1 \times \sin.(C - 90°) = \tan.(a - 90°).\tan.(b - 90°)$ the same result as [*a*].

Examples of the solution of right-angled spherical triangles.

Example 1.

Given $\begin{cases} A = 23° 27' 42'' \\ b = 10\ 39\ 40 \end{cases}$ Required $\begin{cases} c \\ a \\ B \end{cases}$



c determined.

$90° - A$ middle part, $90° - c$, b, adjacent parts;

$\therefore r.\cos. A = \cot. c.\tan. b$.....[1st rule]

$\therefore \log.\cot. c = \log. r + \log.\cos. A - \log.\tan. b$

$\log. r = 10$;

$\therefore \log. r + \log.\cos. 23° 27' 42''$.....$= 19.9625238$

$\log.\tan. 10\ 39\ 40$.....$= 9.2747329$

$\log.\cot. c$.....$= 10.6877909$

$\therefore c = 11° 35' 49''$,

a determined.

b middle part, $90° - A$, a, adjacent parts;
$\therefore r \sin. b = \cot. A . \tan. a$ [1st rule, p, 136,]
$\therefore \log. \tan. a = \log. r + \log. \sin. b - \log. \cot. A$

10 + log. sin. 10° 39′ 40″.....=19.2671709
log. cot. 23 27 42.....=10.3624932
$\therefore$ log. tan. a.....= 8.9046777
$\therefore a = 4° 35′ 26″$.

B determined.

$90° - B = M$, $90° - A$, b opposite parts;
$\therefore r . \cos. B = \sin. A . \cos. b$.....[2d rule]
$\therefore \log. \cos. B + \log. r = \log. \sin. A + \log. \cos. b$

log. sin. 23° 27′ 42″.....= 9.6000308
log. cos. 10 39 40.....= 9.9924380
$\therefore$ log. cos. B + 10.....=19.5924688
$\therefore B = 66° 58′ 1″$.

The above is the solution of a Problem in Astronomy, in which from the obliquity A, and the right ascension b as given quantities, the longitude c of the Sun, and the declination a are required.

Example 2.

Given $\left\{\begin{matrix} a = 27° . 48′ \\ c = 71 . 39 . 37 \end{matrix}\right\}$ Required $\left\{\begin{matrix} b \\ B \end{matrix}\right\}$

b determined.

$90 - c = M$, a, b, opposite parts.

$\therefore r . \cos. c = \cos. a . \cos. b$.....2d rule;
$\therefore \log. \cos. b = 10 + \log. \cos. c - \log. \cos. a$,

10 + log. cos. 71° 39′ 37″.....=19.4978286
log. cos. 27 48 0.....9.9467376
$\therefore$ log. cos. b.....= 9.5510910
and $b = 69° 9′ 48″$.

B determined.

$90 - B = M$, $90 - c$, a, adjacent parts;

$\therefore r \cdot \cos. B = \cot. c \cdot \tan. a$ .....by 1st rule

$\log. \cos. B + 10 = \log. \cot. c + \log. \tan. a,$

$\log. \cot. 71° 39' 37''$ ..... $= 9.5204674$

$\log. \tan. 27\ 48\ 0$ ..... $= 9.7220085$

$10 + \log. \cos. B$ ..... $= 19.2424759$

$\therefore B = 79° 56' 4''.$

This is the solution of an Astronomical Problem, in which, from the latitude of a place $= 90° - 27° 48' = 62° 12'$, and the altitude of the Sun at six o'clock $= 90° - 71° 39' 37'' = 18° 20' 23''$, it is required to find the Sun's declination, which, by the result, would be $90° - 69° 9' 48'' = 20° 50' 12''$; the angle $B = 79° 56' 11''$, in the same Problem, is the Sun's azimuth.

Example 3.

C not a right angle, but c, the side opposite to it, $= 90°$,

Given $\begin{cases} a = 32° 57' 6'' \\ b = 66\ 32\ 0 \end{cases}$ Required $\begin{cases} B \\ C \end{cases}$

C determined.

By the expression [a], p. 129,

$$\cos. C = \frac{\cos. c - \cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}$$

$$= - \frac{\cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}, \text{ since } \cos. c = 0$$

$$= - \cot. a \cdot \cot. b;$$

$\therefore$ in logarithms,

$$\log. \cos. C = \log. \cot. a + \log. \cot. b - 10.$$

Now $- 10 + \log. \cot. 32° 57' 6''$ ..... $= .1882850$

$\log. \cot. 66\ 32\ 0$ ..... $= 9.6376106$

$\therefore \cos. C$ ..... $= 9.8258956$

$\therefore C = 180° - (47° 57' 16'') = 132° 2' 44''$

The supplement of $47° 57' 16''$ is taken, since from the expression

$$\cos. C = - \cot. a \cdot \cot. b$$

$\cos. C$ is negative; $\therefore C > 90°$.

B determined.

By the expression [b], page 129,

$$\cos. B = \frac{\cos. b - \cos. a \,.\, \cos. c}{\sin. a \,.\, \sin. c}$$

$$= \frac{\cos. b}{\sin. a}, \text{ since } \cos. c = 0;$$

$$\therefore \log. \cos. B = 10 + \log. \cos. b - \log. \sin. a$$

$10 + \log. \cos. 66° 32' 0'' \ldots\ldots = 19.6001181$

$\log. \sin. 32\ 57\ 6 \ldots\ldots = 9.7355441$

$\therefore \log. \cos. B \ldots\ldots = 9.8645740$

$$\therefore B = 42° 56' 12''.$$

Here, the Problem is most simply resolved from the fundamental expressions for the cosines of the angles: and there is no need that we recur to the supplemental triangle: if however we do, then since its two angles are $180° - a$, $180° - b$, and the adjacent side $= 180° - C$, by Naper's first Rule, $r \sin. (C - 90°) = \tan. (a - 90°) \tan. (b - 90°)$ or $- r \,.\, \cos. C = \cot. a \,.\, \cot. b$, as before; and similarly for B.

The above is the solution of an Astronomical Problem, in which, from the latitude $= 90° - 32° 57' 6'' = 57° 2' 54''$, and the Sun's declination $= 90° - 66° 32' = 23° 28'$, it is required to find the time at which the Sun rises.

In the solution of plane triangles, one of the cases, see p. 73, is ambiguous: in spherical right-angled triangles, there are three ambiguous cases: and, these are when the quantities, a, c, A are to be found from b and B given: now, by Naper's Rules, a, c, A are given by these three equations;

$$r \,.\, \sin. a = \tan. b \,.\, \cot. B$$
$$r \,.\, \sin. b = \sin. c \,.\, \sin. B;$$
$$r \,.\, \cos. B = \sin. A \,.\, \cos. b;$$

but after sin. a, sin. c, sin. A, are deduced from these equations, there is nothing to determine us, whether we ought to take a, c, A, or $180° - a$, $180° - c$, $180° - A$, ought to be taken; for, the sines of the 3 latter quantities are the same as the sines of the 3

former: and it is easily shewn, that there are two right-angled spherical triangles, which have an angle and side opposite the same in both, but in which the remaining sides, and the remaining angle of the one, are respectively the supplements of the remaining sides and the remaining angle of the other.

The other cases are not ambiguous, and, Naper's Rules, with an attention to the signs of the quantities involved, will enable us to remove the ambiguity which some of these cases appear to have: thus, if b be the middle part, $90° - c$, and $90° - B$, the opposite parts, then sin. b = sin. c. sin. B: if b be required from this equation, will it not be doubtful, whether b or $180° - b$, [since sin. b = sin. $(180° - b)$] ought to be taken? The ambiguity is removed by this property, that, if B be $>$ or $< 90°$, b is $>$ or $< 90°$: for by Naper's 1st Rule, sin. a = cot. B. tan. b; now, sin. a is positive when a is between 0 and 180°, and if B be $> 90°$, cot. B is negative; and consequently, tan. $b = \frac{\text{sin. } a}{\text{cot. } B}$, is negative, and $b > 90°$; if $B < 90°$, cot. B is positive; $\therefore$ tan. $b = \frac{\text{sin. } a}{\text{cot. } B}$ is positive: and $b < 90°$;

similarly, make $90° - B = M$, and $90° - c$, and a adjacent parts, then, cos. B = cot. c. tan. a; $\therefore$ if c be sought,

tan. $c = \frac{\text{tan. } a}{\text{cos. } B}$; if a be $> 90°$, and $B > 90°$, then tan. a and cos. B are both negative; $\therefore$ tan. c is positive, and consequently, c is $< 90°$: if $a > 90°$, and $B < 90°$, or, if $a < 90°$, and $B > 90°$, then $\frac{\text{tan. } a}{\text{cos. } B}$ is negative; $\therefore$ tan. c is negative; $\therefore c > 90°$.

These considerations are so simple and so easily made, that it is, perhaps, better to let the Student endeavour to avail himself of similar ones, than to burthen his memory with the terms and results of formal Propositions: for it must be noticed, that in order to prevent the ambiguity of solutions in right-angled triangles, terms have been invented and Propositions framed relative to the *affections* of the sides and angles: sides and their opposite angles being said to have the *same affection*, when each

is less or greater than 90°; See Simson's *Euclid*, Prop. 13, p. 500, 8vo. edit. 1781.

It has been already proved, that sides and the opposite angles are each greater or less than 90°, that is, have the same affection: again, since by the form [1], p. 133,

$$\cos. c = \cos. a . \cos. b,$$

if a and b be both $> 90°$, cos. a and cos. b are both negative; $\therefore$ their product, which $=$ cos. c, is positive, and $c < 90°$; $\therefore$ if both be $< 90°$, cos. c is positive, and $c < 90°$: if a be $> 90°$, and $b < 90°$, or if $a < 90°$, and $b > 90°$, the product cos. a . cos. b is negative; $\therefore$ cos. c is negative; $\therefore$ c is $> 90°$. This may be easily translated into the terms and language which Robert Simson uses in his Trigonometry. See Prop. 14, *Spherical Trigonometry* at the end of Euclid's *Elements*, p. 500.

The several cases of right-angled spherical triangles being now solved, we will proceed in the next Chapter to the solution of oblique-angled triangles.

CHAP. XI.

Equations exhibiting the Relations of the Sides and Angles of Oblique-angled Spherical Triangles.—Formulæ of Solution deduced from such Equations.—Examples, &c.

In the cases of oblique-angled spherical triangles six quantities are concerned, a, b, c, A, B, C: and the general problem requires us to determine three of the six by means of the three others. We must have equations then between four of these quantities combined all possible ways; but the number of the combinations of six quantities, taken four and four, equals $\frac{6.5.4.3}{1.2.3.4}$, or 15. These combinations are

$$(a\,b\,c\,A),\ (a\,b\,c\,B),\ (a\,b\,c\,C)$$
$$(A\,B\,C\,a),\ (A\,B\,C\,b),\ (A\,B\,C\,c)$$
$$(a\,C\,b\,A),\ (a\,C\,b\,B),\ (a\,B\,c\,A),\ (a\,B\,c\,C),\ (b\,A\,c\,B),\ (b\,A\,c\,C)$$
$$(a\,A\,b\,B),\ (a\,C\,c\,A),\ (b\,C\,c\,B).$$

Now, the number of combinations essentially different is the number of the horizontal rows, or four: for instance, the combinations of the first row depend on three similar equations:

$$\text{cos. } A = \frac{\text{cos. } a - \text{cos. } b \,.\, \text{cos. } c}{\text{sin. } b \,.\, \text{sin. } c} \quad \ldots\ldots[1]$$

$$\text{cos. } B = \frac{\text{cos. } b - \text{cos. } a \,.\, \text{cos. } c}{\text{sin. } a \,.\, \text{sin. } c} \quad \ldots\ldots[2]$$

$$\text{cos. } C = \frac{\text{cos. } c - \text{cos. } a \,.\, \text{cos. } b}{\text{sin. } a \,.\, \text{sin. } b} \quad \ldots\ldots[3].$$

The combinations of the fourth row depend on three similar equations,

$$\frac{\text{sin. } a}{\text{sin. } b} = \frac{\text{sin. } A}{\text{sin. } B};\ \frac{\text{sin. } a}{\text{sin. } c} = \frac{\text{sin. } A}{\text{sin. } C};\ \frac{\text{sin. } b}{\text{sin. } c} = \frac{\text{sin. } B}{\text{sin. } C},$$

and similarly for the remaining two rows: hence the solution of all

the cases of oblique-angled triangles is reduced, in fact, to four equations, and these four equations must be deduced, as the equations in p. 134 were, by the ordinary processes of substitution and elimination.

We will now proceed to deduce these four equations.

First equation belonging to the form $(a\,b\,c\,A)$,

$$\cos. A = \frac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c}.$$

Second equation belonging to $(A\,B\,C\,a)$.

In order to obtain this, eliminate cos. b, cos. c, sin. b, sin. c from the equations [1], [2], [3]: or, more simply to obtain it, take the supplemental triangle; then, if a', b', c' are its sides, A', B', C', its angles, we have, by the form [1],

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b' \,.\, \sin. c'};$$

but
$$\begin{aligned}
\cos. A' &= \cos. (180^\circ - a) = -\cos. a \\
\cos. a' &= \cos. (180^\circ - A) = -\cos. A \\
\cos. b' &= \cos. (180^\circ - B) = -\cos. B \\
\cos. c' &= \cos. (180^\circ - C) = -\cos. C \\
\sin. b' &= \sin. (180^\circ - B) = \sin. B \\
\sin. c' &= \sin. (180^\circ - C) = \sin. C;
\end{aligned}$$

$$\text{consequently, } -\cos. a = \frac{-\cos. A - \cos. B \,.\, \cos. C}{\sin. B \,.\, \sin. C};$$

$$\text{or, } \cos. a = \frac{\cos. A + \cos. B \,.\, \cos. C}{\sin. B \,.\, \sin. C}.$$

Third equation belonging to $(a\,C\,b\,A)$.

In order to obtain this, substitute in the equation [1], instead of cos. c, its value derived from the equation [3]: and instead of sin. c, substitute $\frac{\sin. C}{\sin. A} \,.\, \sin. a$,

$$\text{then, } \cos. A \,.\, \sin. b \,.\, \frac{\sin. C}{\sin. A} \,.\, \sin. a =$$

$$\cos. a - \cos. b (\cos. C . \sin. a . \sin. b + \cos. a . \cos. b)$$
$$= \cos. a \sin.^2 b - \cos. C . \sin. a . \sin. b . \cos. b.$$

Hence, on dividing each side of the equation by $\sin. a \times \sin. b$, there results

$$\cot. A . \sin. C = \cot. a . \sin. b - \cos. C . \cos. b.$$

Fourth equation belonging to $(a\,A\,b\,B)$.

This equation is deduced in Cor. 2. to Prop. 16, where it is proved, that $\frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B}$.

These four equations analytically resolve the Problem; or, by means of them, any three quantities being given, the fourth may be found; but it is plain from their inspection, that they do not afford convenient solutions, since none of them are under a form adapted to logarithmic computation; and even, if, in order to find one of the quantities involved in the equation, we were to express the equation under a form adapted to a logarithmic computation, such modified form would be useless, except in the case for which it was contrived: that is, would be useless, if one of the quantities, by which the required quantity was expressed, should itself be required to be computed, the previously required quantity becoming, in this second case, one of the given quantities: for instance, in the combination $(a\,b\,c\,A)$ if A be the quantity sought, then, by p. 130,

$$1 + \cos. A, \text{ or, } 2 \cos.^2 \frac{A}{2} =$$
$$\frac{2}{\sin. b . \sin. c} \left\{ \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{a+b+c}{2} - a\right) \right\}$$

whence, by a logarithmic computation, $\cos. A$ and A can be found; but, from such form, if A were given, and a required, a cannot be immediately and conveniently found: and, on this account, something more is required of the analyst, than mere equations that exhibit the possibility of solutions: he ought to furnish formulæ, from which, the quantity, whatever it be, side or angle, may by a direct, certain, and commodious process, be found. Formulæ then, as it will easily be seen, by which one

quantity may be computed from others that are given, must be more in number than the equations which merely exhibit the relations of the quantities. This does not take place in right-angled spherical triangles; in which the formulæ of solution need not exceed in number the equations: for instance, the equation cos. c = cos. a . cos. b represents the relation between c, a, b; and from the same equation may be found, by a like logarithmic computation, either c from a and b; or a from b and c; or b from a and c; the same may be said of the second equation, that is, cos. A = cot. c . tan. b: either A, or c, or b, may, from the equation as it stands, be found with equal facility.

The solution of oblique-angled spherical triangles will be found, by what follows, to require six cases; and in using the foregoing forms of combination such as ($a\,b\,c\,A$), the quantity sought will be placed last.

Cases of Oblique-angled Spherical Triangles.

Prop. 18. Case 1st, ($a\,b\,c\,A$)

The sides a, b, c, are given, and the angle A is required.

First Method of Solution.

By Prob. 16, page 130, if $S = \frac{a+b+c}{2}$,

$$\sin. A = 2 . \frac{\sqrt{\{\sin. S . \sin. (S-a) . \sin. (S-b) \sin. (S-c)\}}}{\sin. b . \sin. c}$$

and, log. sin. $A = \frac{1}{2}\{20 + 2 \log. 2 + \log. \sin. S + \log. \sin. (S-a) + \log. \sin. (S-b) + \log. \sin. (S-c)\} - \log. \sin. b - \log. \sin. c.$

In order to find log. sin. B, subtract, in the above form, instead of log. sin. b and log. sin. c, log. sin. a and log. sin. c; and, to find log. sin. C, subtract, log. sin. a, and log. sin. b.

Second Method.

By Proposition 16, page 130,

$$1+\cos. A = 2\frac{\sin. S . \sin. (S-a)}{\sin. b . \sin. c}; \therefore \cos.^2 \frac{A}{2} = \frac{\sin. S . \sin. (S-a)}{\sin. b . \sin. c}$$

and $\log. \cos. \frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log. \sin. S + \log. \sin. (S-a) - \log. \sin. b - \log. \sin. c\}$$

Third Method.

By the same Proposition, in the same page,

$$1 - \cos. A =$$

$$2\frac{\sin. (S-b) \sin. (S-c)}{\sin. b . \sin. c}; \therefore \sin.^2 \frac{A}{2} = \frac{\sin. (S-b) \sin. (S-c)}{\sin. b . \sin. c}$$

and $\log. \sin. \frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log. \sin. (S-b)+\log. \sin. (S-c)-\log. \sin. b-\log. \sin. c\}$$

Fourth Method.

Divide the 3d equation by the 2d, and

$$\frac{\sin. \frac{A}{2}}{\cos. \frac{A}{2}}, \text{ or } \tan. \frac{A}{2} = \sqrt{\left(\frac{\sin. (S-b) \sin. (S-c)}{\sin. S . \sin. (S-a)}\right)}$$

and, $\log. \tan. \frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log. \sin. (S-b)+\log. \sin. (S-c)-\log. \sin. S-\log. \sin. (S-a)\}$$

since $\sin. b = \frac{r^2}{\text{cosec. } b}$, $\log. \sin. b = 20 - \log. \text{cosec. } b$;

$\therefore$ instead of subtracting log. sin. b, &c. in the above forms, we may (which with certain Tables is a convenience) add log. cosec. $b - 20$, &c.

Example.

$a = 50^\circ\ 54'\ 32''$ Numerator computed.
$b = 37\ \ 47\ \ 18$
$c = 74\ \ 51\ \ 50$ $29 + 2 \log. 2 = 20.6020599$
$163\ \ 33\ \ 40$

$S = \frac{a+b+c}{2} = 81^\circ\ 46'\ 50''$ ..... sin. $= 9.9955158$

$S - a$ ..... $= 30\ \ 52\ \ 18$ ..... sin. $= 9.7102163$
$S - b$ ..... $= 43\ \ 59\ \ 32$ ..... sin. $= 9.8417102$
$S - c$ ..... $= 6\ \ 55\ \ 0$ ..... sin. $= 9.0807189$

$2 \mid 59.2302211$
$29.6151105\ [a]$

B determined.	*A* determined.
log. sin. $50^\circ\ 54'\ 32'' = 9.8899424$	log. sin. $37^\circ\ 47'\ 18'' = 9.7872806$
log. sin. $74\ \ 51\ \ 50 = 9.9846660$	log. sin. $74\ \ 51\ \ 50 = 9.9846660$
$[c]\ 19.8746084$	$19.7719466\ [b]$
$[a] - [c] = 9.7405021$	$[a] - [b] = 9.8431639$
$\therefore B = 33^\circ\ 22'\ 45''$	$\therefore A = 44^\circ\ 10'\ 40''.$

C determined.

log. sin. $50^\circ\ 54'\ 32'' = 9.8899424$
log. sin. $37\ \ 47\ \ 18 = 9.7872806$
$[m]\ 19.6772230$
$[a] - [m] = 9.9378875$

$\therefore C = 180^\circ - \{60^\circ\ 4'\ 54''\}$
$= 119\ \ 55\ \ 6,$

C is greater than 90°, since

$$\cos. C = \frac{\cos. c - \cos. a \,.\, \cos. b}{\sin. a \,.\, \sin. b}$$

which is negative, since $\cos. c < \cos. a \cos. b$.

Solution of the former Example by the second Method.

$$20 = 20$$

$S = 81^\circ\ 46'\ 50''$.....sin. $= 9.9955158$

$S - a = 30\ 52\ 18$.....sin. $= 9.7102163$

39.7057321 [d]

$b = 37^\circ\ 47'\ 18''$.....sin. $= 9.7872806$

$c = 74\ 51\ 50$.....sin. $= 9.9846660$

19.7719466.....[b]

$$[d] - [b] = 19.9337855$$

$$\tfrac{1}{2}([d] - [b]) = 9.9668927 = \log.\cos.\frac{A}{2};$$

$$\therefore \frac{A}{2} = 22^\circ\ 5'\ 20'',$$

and $A = 44\ 10\ 40$, as before.

By the third Method.

$$20 = 20$$

$S - b = 43^\circ\ 59'\ 32''$.....sin. $= 9.8417102$

$S - c = 6\ 55\ 0$.....sin. $= 9.0807189$

38.9224291.....[f]

$$[b] = 19.7719466$$

$$[f] - [b] = 19.1504825$$

$$\tfrac{1}{2}([f] - [b]) = 9.5752412 = \log.\sin.\frac{A}{2}$$

$$\therefore \frac{A}{2} = 22^\circ\ 5'\ 20'',\ \text{and}\ A = 44^\circ\ 10'\ 40'',\ \text{as before.}$$

By the fourth Method.

$$20 = 20$$

log. sin. $(S - b)$ + log. sin. $(S - c)$, by last, or [f] $= 38.9224291$

log. sin. S + log. sin. $(S - a)$, by 2d solution, or [d] $= 39.7057321$

19.2166970

$$\therefore \log.\tan.\frac{A}{2} = 9.6083485$$

$$\therefore \frac{A}{2} = 22^\circ\ 5'\ 20'',\ \text{and}\ A = 44^\circ\ 10'\ 40'',\ \text{as before.}$$

The solution of this Example becomes the solution of an Astronomical Problem, when from the co-latitude of the place b, or 37° 47′ 18″, the co-declination of the Sun c, or 74° 51′ 50″, and the zenith distance a, or 50° 54′ 32″: the time from noon or the angle A is required; the angle C is the azimuth.

Any one of the four preceding methods may be used, but not, in point of brevity, with equal advantage: if one angle only be required, the shortest solution is plainly by means of one of the three last formulæ: but, if all the angles of the spherical triangle be required, then the first method is as short, and quite as convenient, as the three last methods *.

Any one of these four methods may be used, but not, with regard to numerical accuracy, in all cases, with equal advantage. If the angle sought, A for instance, should happen to be nearly 90°, then the first method is to be superseded by one of the three latter, and this for reasons precisely the same, as those which have been stated in page 85, to which the Reader is referred.

Case 2d, $(ABCa)$.

The angles A, B, C, are given, and the side a required.

First Method of Solution.

Let the sides and angles of the supplemental or polar triangle be a', b', c'; A', B', C': then, by the last Case,

$$\sin. A' = 2 \frac{\sqrt{\{\sin. S . \sin. (S - a') . \sin. (S - b') \sin. (S - c')\}}}{\sin. b' . \sin. c'}$$

* In the *Logarithmical Arithmetike*, published in 1631, the first case is solved by the second method, and in Vlacq's Tables, published at Gouda in 1633, the year in which the Trigonometria Britannica was published, the solution is by the third method; the rules of solution, then, 200 years ago, were as plain and precise as they now are: yet, in the mode of proof we have gained something, which is certainly more plain and direct than Vlacq's.

but $S = \frac{a' + b' + c'}{2} = \frac{1}{2}(180° - A + 180° - B + 180° - C) =$

$270° - \frac{A + B + C}{2} = (270° - S')$ if, $\frac{A + B + C}{2} = S'$;

hence, sin. S = − cos. S'.

Again, $S - a' = 270° - S' - (180° - A) = 90° - (S' - A)$

$\therefore$ sin. $(S - a')$ = cos. $(S' - A)$;

also, $S - b' = 90° - (S' - B)$; $\therefore$ sin. $(S - b')$ = cos. $(S' - B)$; and similarly, sin. $(S - c')$ = cos. $(S' - C)$.

Again, sin. b' = sin. $(180° - B)$ = sin. B; sin. c' = sin. C, and sin. A' = sin. a.

Hence,

$$\text{sin. } a = 2\sqrt{\{-\cos. S' . \cos. (S' - A) . \cos. (S' - B) . \cos. (S' - C)\}} \over \text{sin. } B . \text{sin. } C$$

and in logarithms,

log. sin. $a = \frac{1}{2}\{2 \log. 2 + 20 + \log. \cos. S' + \log. \cos. (S' - A)$

$+ \log. \cos. (S' - B) + \log. \cos. (S' - C)\} - (\log. \sin. B + \log. \sin. C)$

and sin. b, sin. c, are represented by fractions that have the same numerators as sin. a, and denominators, which are equal to sin. A . sin. C, and sin. A . sin. B, respectively.

The cos. S' under the vinculum is affected with the negative sign. Now, by Prop. 9, page 121, $A + B + C > 180°$, and $< 540°$; $\therefore \frac{A + B + C}{2} > 90°$, and $< 270°$; $\therefore$ cos. S' is negative, or − cos. S' is positive. Again, $S' - A = \frac{B + C - A}{2}$: but, by Prop. 2, page 117, $b' + c' > a'$; $\therefore 180° - B + 180° - C > 180° - A$; $\therefore B + C - A < 180°$, and $\frac{B + C - A}{2} < 90°$; $\therefore$ cos. $(S' - A)$ is positive; so are, by similar proofs, cos. $(S' - B)$ and cos. $(S' - C)$.

Hence, in the foregoing expression for sin. $\frac{a}{2}$, the quantity under the radical sign is really a positive quantity.

As the expression for sin. A, has, by the aid of the supplemental triangle, been employed to represent a side in terms of the angles, so, in like manner, may the expressions for cos. $\frac{A}{2}$, sin. $\frac{A}{2}$, tan. $\frac{A}{2}$; thus by the expression, page 149,

$$\cos.\frac{A'}{2}=\sqrt{\left(\frac{\sin.S.\sin.(S-a')}{\sin.b'.\sin.c'}\right)},$$

or, since $\cos.\frac{A'}{2}=\cos.\frac{(180°-a)}{2}=\sin.\frac{a}{2}$, we shall have a

Second Method of Solution,

And, a may be found from the expression,

$$\sin.\frac{a}{2}=\sqrt{\left(\frac{-\cos.S'.\cos.(S'-A)}{\sin.B.\sin.C}\right)};$$

and similarly, a

Third Method,

And, a may be found from the expression,

$$\cos.\frac{a}{2}=\sqrt{\left(\frac{\cos.(S'-B).\cos.(S'-C)}{\sin.B.\sin.C}\right)};$$

and similarly, a

Fourth Method,

And, a may be found from the expression,

$$\tan.\frac{a}{2}=\sqrt{\left(\frac{-\cos.S'.\cos.(S'-A)}{\cos.(S'-B).\cos.(S'-C)}\right)}.$$

Example by the first Method.

a determined.

$A = 44° 10' 40''$ ..... sin. = 9.8431624 ..... [1]
$B = 33\ 22\ 45$ ..... sin. = 9.7405025 ..... [2]
$C = 119\ 55\ 6$ ..... sin. = 9.9378874 ..... [3]
197 28 31
$\therefore S' = 98\ 44\ 15.5$ ..... cos. = 9.1815867
$S' - A = 54\ 33\ 35.5$ ..... cos. = 9.7633172
$S' - B = 65\ 21\ 30.5$ ..... cos. = 9.6200732
$S' - C = -(21\ 10\ 50.5)$ ..... cos. = 9.9696235
20 + 2 log. 2 = ... 20.6020599
2)59.1366605 ..... [s]
29.5683302
[2] + [3] = 19.6783899
log. sin. a = 9.8899403
and $a = 50° 54' 30''.8$.

b determined.

[s] ..... = 29.5683302
[1] + [3] ..... = 19.7810498
sin. b ..... = 9.7872804
and, b ..... = 37° 47' 18".

c determined.

[s] ..... = 29.5683302
[1] + [2] ..... = 19.5836649
sin. c ..... = 9.9846653
and c ..... = 74° 51' 49"

By the Second Method.

log. r^2 ..... = 20
$S' = 98° 44' 15''.5$ ..... cos. = 9.1815867
$S' - A = 54\ 33\ 35.5$ ..... cos. = 9.7633172
38.9449039
[2] + [3], or log. sin. B + log. sin. C ..... = 19.6783899
2)19.2665140
$\therefore$ log. sin. $\frac{a}{2}$ = 9.6332570

$\therefore \frac{a}{2} = 25° 27' 15''\frac{1}{2}$
and $a = 50\ 54\ 30\frac{1}{2}$.

By the Third Method.

$\log. r^2 \ldots\ldots = 20$

$S' - B = 65°\ 21'\ 30''.5 \ldots\ldots \cos. = 9.6200732$

$S' - C = -\ 21\ \ 10\ \ 50.5 \ldots\ldots \cos. = 9.9696235$

39.5896967

$[2] + [3] \ldots\ldots 19.6783899$

$2)19.9113068$

$\therefore \log. \cos. \frac{a}{2} = 9.9556534$

$\therefore \frac{a}{2} = 25°\ 27'\ 15''\frac{2}{5},$

and $a = 50\ \ 54\ \ 30\frac{4}{5}.$

By the Fourth Method.

$\log. r^2 \ldots\ldots = 20$

$\log. \cos. S' + \log. \cos. (S' - A) \ldots\ldots = 18.9449039$

$\log. \cos. (S' - B) + \log. \cos. (S' - C) \ldots\ldots = 19.5896967$

$2)19.3552072$

$\log. \tan. \frac{a}{2} = 9.6776036$

$\therefore \frac{a}{2} = 25°\ 27'\ 15''\frac{2}{5},$

and $a = 50\ \ 54\ \ 30\frac{4}{5}.$

CASE 3d. $(a\,C\,b\,A)$.

a, b, two sides and C the included angle are given, and the angle A is required.

By COR. 2. Prop. 16, page 131,

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}; \quad \therefore \frac{\sin. A \pm \sin. B}{\sin. B} = \frac{\sin. a \pm \sin. b}{\sin. b}.$$

By COR. to Prop. 15. $\cos. A + \cos. B = 2 \cdot \frac{\sin. (a+b)}{\sin. c} \sin.^2 \frac{C}{2}$

$$\therefore \frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. a + \sin. b}{2 . \sin. (a+b) \sin.^2 \frac{C}{2}} \cdot \frac{\sin. c . \sin. B}{\sin. b};$$

but $\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \left(\frac{A+B}{2}\right)$ page 28, and

$\frac{\sin. c . \sin. B}{\sin. b} = \sin. C = 2 \sin. \frac{C}{2} \times \cos. \frac{C}{2}$, p. 13 and 36.

$$\therefore \tan. \left(\frac{A+B}{2}\right) = \frac{\sin. a + \sin. b}{2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} \cdot \cot. \frac{C}{2}$$

$$= \frac{2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a-b}{2}\right)}{2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} \cot. \frac{C}{2} \text{ (by form [5], page 27)}$$

$$\text{and } \therefore \tan. \left(\frac{A+B}{2}\right) = \frac{\cos. \left(\frac{a-b}{2}\right)}{\cos. \left(\frac{a+b}{2}\right)} \cdot \cot. \frac{C}{2} \ldots\ldots\ldots\ldots\ldots [1]$$

Similarly,

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. a - \sin. b}{2 . \sin. (a+b) . \sin.^2 \frac{C}{2}} \cdot \sin. C,$$

$$= \frac{2 . \sin. \left(\frac{a-b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)}{2 . 2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} 2 \cot. \frac{C}{2};$$

$$\text{or, } \tan. \left(\frac{A-B}{2}\right) = \frac{\sin. \left(\frac{a-b}{2}\right)}{\sin. \left(\frac{a+b}{2}\right)} \cdot \cot. \frac{C}{2} \ldots\ldots\ldots\ldots [2]$$

Hence, since by the formula [1], $\tan. \left(\frac{A+B}{2}\right)$ is determined, $\frac{A+B}{2}$ becomes known, let it $= S$: and, by the formula [2],

tan. $\left(\frac{A-B}{2}\right)$ is determined, and $\frac{A-B}{2}$ becomes known, let it equal D, then, since

$$\frac{A+B}{2} = S$$

$$\frac{A-B}{2} = D;$$

by addition, $A = S + D$,
by subtraction, $B = S - D$.

Example.

$C = 36° \ 45' \ 28''$

$a = 84 \ \ 14 \ \ 29$

$b = 44 \ \ 13 \ \ 45$

$A + B$ determined.

$\frac{C}{2} = 18° \ 22' \ 44''$..cot. $= 10.4785395$

$\frac{a-b}{2} = 20 \ \ 0 \ \ 22$..cos. $= 9.9729690$

20.4515085

$\frac{a+b}{2} = 64 \ \ 14 \ \ 7$..cos. $= 9.6381663$

$\therefore$ log tan. $\frac{A+B}{2} = 10.8133422$

$\therefore \frac{A+B}{2} = 81° \ 15' \ 44''.41.$

$A - B$ determined.

$\frac{C}{2} = 18° \ 22' \ 44''$.....cot. $= 10.4785395$

$\frac{a-b}{2} = 20 \ \ 0 \ \ 22$.....,sin. $= 9.5341789$

20.0127184

$\frac{a+b}{2} = 64 \ \ 14 \ \ 7$.....sin. $= 9.9545255$

10.0581929

$\frac{A-B}{2} = 48° \ 49' \ 36''$

A and B determined.	c determined.
$\frac{A+B}{2} = 81° 15' 44''.41$	sin. $36° 45' 28'' = 9.7770158$
$\frac{A-B}{2} = 48 \ 49 \ 38$	sin. $44 \ 13 \ 45 = 9.8435629$
$\therefore A = 130° 5' 22''.41$	19.6205787
$B = 32 \ 26 \ 6.41$	sin. $32 \ 26 \ 6 = 9.7294422$
	$\therefore$ log. sin. $c = 9.8911365$
	$\therefore c = 51° 6' 12''$.

The expressions for tan. $\left(\frac{A+B}{2}\right)$ and tan. $\left(\frac{A-B}{2}\right)$, expanded into a proportion, are called, from their Inventor, Naper's *Analogies*.

The angles A and B being determined by the above forms, the side c may be determined, as it has been in the foregoing Example, from the expression $\frac{\text{sin. } c}{\text{sin. } C} = \frac{\text{sin. } b}{\text{sin. } B}$: but, it may be desirable, as in the corresponding case of rectilinear triangles (see page 71), to determine c immediately without the intervening process of finding the angles A and B: and, in fact, many Problems in Astronomy*, require from the data of two sides and the included angle, solely the determination of the opposite side c.

Determination of the side c.

Second Method.

Since cos. $C = \frac{\text{cos. } c - \text{cos. } a \text{ . cos. } b}{\text{sin. } a \text{ . sin. } b}$, page 129,

$\therefore$ cos. c = cos. a. cos. b + sin. a. sin. b. cos. C;

but cos. $C = 1 -$ ver. sin. C (ver. sin. stands for versed sine);

* For instance, in finding the Moon's distance from a Star; in deducing the altitude of a Star from the latitude, declination and hour-angle (two Problems useful in determining the Longitude); in deducing, in the case of an occultation, the Moon's distance from a Star; in determining the altitude of the nonagesimal (see *Astronomy*, p. 364.): in determining the latitude from 2 altitudes and the time between, (see *Astronomy*, p. 422.), &c.

$$\therefore \cos. c = \cos. a. \cos. b + \sin. a. \sin. b - \sin. a. \sin. b. \text{ver. sin. } C$$
$$= \cos. (a - b) - \sin. a. \sin. b. \text{ver. sin. } C;$$

$$\therefore 1 - \cos. c, \text{ or, } 2 \sin.^2 \frac{c}{2} = \text{ver. sin. } (a-b) + \sin. a. \sin. b. \text{ver. sin. } C$$
$$= \text{ver. sin. } (a-b) \left(1 + \frac{\sin. a. \sin. b. \text{ver. sin. } C}{\text{ver. sin. } (a-b)}\right)$$

Assume * $\tan.^2 \theta = \dfrac{\sin. a. \sin. b. \text{ver. sin. } C}{\text{ver. sin. } (a-b)}$;

and in logarithms, $2 \log. \tan. \theta =$
$\log. \sin. a + \log. \sin. b + \log. \text{ver. sin. } C - \log. \text{ver. sin. } (a-b)$ [p]

then $2 \sin.^2 \dfrac{C}{2} = \text{ver. sin. } (a-b). \sec.^2 \theta$, and

$\log. 2 + 2 \log. \sin. \dfrac{c}{2} = \log. \text{ver. sin. } (a-b) + 2 \log. \sec. \theta - 10$ [q]

Former *Example.*

c computed independently of A and B by 2d Method.

Determination of the subsidiary angle θ by the form [p]

$a = 84°\ 14'\ 29''$	sin.	$= 9.9978028$
$b = 44\ 13\ 45$	sin.	$= 9.8435629$
$C = 36\ 45\ 28$	ver. sin.	$= 9.2984762$
		29.1398419
$a - b = 40\ 0\ 44$	ver. sin.	$= 9.3693878$
$\therefore 2 \log. \tan. \theta$		$= 19.7704541$
and $\log. \tan. \theta$		$= 9.8852270$

* This is the instance to which we alluded in speaking, page 96, of the use of Trigonometrical formulæ in computing log. $(a+b)$.

Determination of c by the form [q], page 160.

2 log. sec. θ ..... $= 20.2012488$

log. ver. sin. $40° \; 0' \; 44''$ ..... $= \underline{9.3693878}$

29.5706366

$10 +$ log. 2 ..... $= \underline{10.3010300}$

$\therefore 2$ log. sin. $\frac{c}{2}$ ..... $= 19.2696066$

and log. sin. $\frac{c}{2}$ ..... $= 9.6348033$

$$\therefore \frac{c}{2} = 25° \; 33' \; 5''\tfrac{8}{11},$$

and $c = 51 \quad 6 \quad 11.5$, nearly.

This is, perhaps, the most commodious form for computing c, for, when we use it, we need not consider whether the fraction $\frac{\text{sin. } a \text{ . sin. } b \text{ . v. sin. } C}{\text{v. sin. } (a - b)}$ is $>$ or $<$ 1, since tan. θ admits of all degrees of magnitude. It is easy, however, to give another formula of computation, thus:

Third Method of computing c.

$$\cos. c = \cos. a \text{ . } \cos. b + \sin. a \text{ . } \sin. b \text{ . } \cos. C$$

$$= \cos. a \text{ . } \cos. b + \sin. a. \sin. b. \left(2 \cos.^2 \frac{C}{2} - 1\right)$$

$$= \cos. (a + b) + \sin. a \text{ . } \sin. b \text{ . } 2 \cos.^2 \frac{C}{2};$$

$$\therefore 1 - 2 \text{ . } \sin.^2 \frac{c}{2} = \cos. (a + b) + \sin. a. \sin. b \text{ . } 2 \cos.^2 \frac{C}{2};$$

and $$\therefore 2 \sin.^2 \frac{c}{2} = 2 \text{ . } \sin.^2 \frac{a+b}{2} - 2 \sin. a. \sin. b. \cos.^2 \frac{C}{2}.$$

Let $$\sin. a \text{ . } \sin. b \text{ . } \cos.^2 \frac{C}{2} = \sin.^2 M;$$

and $$\therefore \text{log. sin. } M = \tfrac{1}{2}\left(2 \text{ log. cos. } \frac{C}{2} + \text{log. sin. } a + \text{log. sin. } b - 20\right)$$

then $$\sin.^2 \frac{c}{2} = \sin.^2 \frac{a + b}{2} - \sin.^2 M$$

$$= \sin.\left(\frac{a}{2}+\frac{b}{2}+M\right).\sin.\left(\frac{a}{2}+\frac{b}{2}-M\right)$$

by the form [*c*], p. 26, and in logarithms,

$$\log.\sin.\frac{c}{2}=\tfrac{1}{2}\left(\log.\sin.\left(\frac{a}{2}+\frac{b}{2}+M\right)+\log.\sin.\left(\frac{a}{2}+\frac{b}{2}-M\right)\right)$$

This is the kind of form which Laplace has employed, Livre 2, *Mecanique Celeste*, page 227.

Former Example, by the third Method.

$\frac{C}{2}=18°\ 22'\ 44\ \ 2.\ \cos.=19.9545254$

$a=84\ \ 14\ \ 29.....\sin.=\ 9.9978028$

$b=44\ \ 13\ \ 45.....\sin.=\ 9.8435629$

$\frac{a+b}{2}=64\ \ 14\ \ 7.....39.7958911$

$M=52\ \ 14\ \ 23..... \therefore\ 9.8979455=\log.\sin.M=\log.\sin.52°\ 14'\ 23''$

$\therefore\ \frac{a+b}{2}+M=116°\ 28'\ 30''\ \sin.=\ 9.9518856$

$\frac{a+b}{2}-M=\ 11\ \ 59\ \ 44\ \sin.=\ 9.3177204$

19.2696060

$9.6348030=\log.\sin.\frac{c}{2};$

$\therefore\ \frac{c}{2}=25°\ 33'\ 5''\frac{22}{44}$

$c=51\ \ 6\ \ 11.33,$ nearly.

Fourth Method.

$$\cos.\ c=\cos.\ a.\cos.\ b+\sin.\ a.\sin.\ b.\cos.\ C$$

$$1-\text{ver. sin.}\ c=\cos.\ a.\cos.\ b+\sin.\ a.\sin.\ b-\sin.\ a.\sin.\ b.\ \text{ver. sin.}\ C;$$

$$\therefore\ \text{ver. sin.}\ c=1-\cos.(a-b)+\sin.\ a.\sin.\ b\ .\ \text{ver. sin.}\ C$$
$$=\text{ver. sin.}\ (a-b)+\sin.\ a.\sin.\ b\ \text{ver. sin.}\ C,$$

which formula, translated into words, becomes the precept given in Sherwin's Table, page 44, (edit. 1771) for finding the side opposite

to the included angle. The Author gives in his Work a Table of natural versed sines, which are plainly necessary in his mode of computation.

If we use one of the preceding formulæ for computing c, we may, if we please, determine A and B without the aid of Naper's *Analogies*, and by these expressions,

$$\text{sin. } A = \frac{\text{sin. } C \cdot \text{sin. } a}{\text{sin. } c}; \quad \text{sin. } B = \frac{\text{sin. } C \cdot \text{sin. } b}{\text{sin. } c}.$$

Case IV. ($A\,c\,B\,a$).*

A, B, two angles and c the adjacent side are given, and the side a is required.

The solution will be deduced from the former, by the aid of the supplemental triangle; A', B', C', a', b', c' being its angles and sides.

$$\text{tan.}\left(\frac{A'+B'}{2}\right) = \text{tan.}\left(\frac{(180°-a)+(180°-b)}{2}\right) = \text{tan.}\left(180° - \frac{a+b}{2}\right)$$

$$= -\text{tan.}\left(\frac{a+b}{2}\right)$$

$$\text{tan.}\left(\frac{A'-B'}{2}\right) = \text{tan.}\left(\frac{b-a}{2}\right) = -\text{tan.}\left(\frac{a-b}{2}\right)$$

$$\text{sin.}\left(\frac{a'+b'}{2}\right) = \text{sin.}\left(180° - \frac{A+B}{2}\right) = \text{sin.}\left(\frac{A+B}{2}\right)$$

[see Cor. 6, Prob. 3.]

$$\text{sin.}\left(\frac{a'-b'}{2}\right) = \text{sin.}\left(\frac{B-A}{2}\right) = -\text{sin.}\left(\frac{A-B}{2}\right);$$

---

* This combination is in the third row (see p. 145,) and therefore not essentially different from the first of that row, namely, ($a\,C\,b\,A$) which has been already considered. In fact, the two equations involving the four quantities are precisely similar: but, from what was said in p. 147, the same formula of solution cannot suit each case, since in the former, an angle is the quantity required, which in the latter, becomes one of the quantities given.

$$\therefore \text{generally, } \sin.\left(\frac{a' \pm b'}{2}\right) = \pm \sin.\left(\frac{A \pm B}{2}\right).$$

Again,

$$\cos.\frac{a'+b'}{2} = \cos.\left(180^\circ - \frac{A+B}{2}\right) = -\cos.\left(\frac{A+B}{2}\right)$$

[see Cor. 6, Prob. 3.]

$$\cos.\left(\frac{a'-b'}{2}\right) = \cos.\left(\frac{B-A}{2}\right); \text{ or } = \cos.\left(\frac{A-B}{2}\right);$$

$$\text{lastly, } \cot.\frac{C'}{2} = \cot.\left(90^\circ - \frac{c}{2}\right) = \tan.\frac{c}{2}.$$

Having now transformed all the terms in [1], [2], (see p. 157), into different expressions, if we substitute in [1], [2], there results

$$\tan.\left(\frac{a+b}{2}\right) = \frac{\left(\cos.\frac{A-B}{2}\right)}{\left(\cos.\frac{A+B}{2}\right)} \cdot \tan.\frac{c}{2} \ldots\ldots\ldots\ldots [3],$$

whence $\frac{a+b}{2}$,

and

$$\tan.\left(\frac{a-b}{2}\right) = \frac{\sin.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A+B}{2}\right)} \tan.\frac{c}{2} \ldots\ldots\ldots\ldots [4],$$

whence $\frac{a-b}{2}$ may be derived.

These equations [3], [4], expanded into a proportion, are called, as the former similar ones [1], [2] were, Naper's *Analogies*.

Example.

$c = 51° \ 6' \ 12''$

$A = 130 \ 5 \ 22$

$B = 32 \ 26 \ 6$

$\frac{A-B}{2} = 48 \ 49 \ 38$ .....cos. = 9.8184449 .....sin. = 9.8766379

$\frac{c}{2} = 25 \ 33 \ 6$ .....tan. = 9.6795082 .....tan. = 9.6795032

19.4979481 19.5561411

$\frac{A+B}{2} = 81 \ 15 \ 44$ .....cos. = 9.1815936 .....sin. = 9.9949301

tan. $\frac{a+b}{2}$ = 10.3163545 | tan. $\frac{a-b}{2}$ = 9.5612110

and $\frac{a+b}{2} = 64° \ 14' \ 6''\frac{1}{8}$ | and $\frac{a-b}{2} = 20° \ 0' \ 22''\frac{2}{13}$

$\frac{a-b}{2} = 20 \ 0 \ 22\frac{2}{13}$

$\therefore a = 84° \ 14' \ 28''\frac{29}{104}$

$b = 44 \ 13 \ 43\frac{101}{104}$.

C determined.

sin. 130° 5′ 22″ ..... = 9.8836842

sin. 51 6 12 ..... = 9.8911357

19.7748199

sin. 84 14 29 ..... = 9.9978028

$\therefore$ log. sin. *C* ..... = 9.7770171

$\therefore C = 36° \ 45' \ 28''$.

a and *b* being determined, *C* may be determined from this expression, sin. C = sin. $A \times \frac{\text{sin. } c}{\text{sin. } a}$, as it was in the preceding Example, or, without the intervening process of finding *a* and *b*, by the following method.

Determination of the angle C.

In the supplemental triangle, by the original form [c], p. 129,

$$\cos. c' = \sin. a' . \sin. b' . \cos. C' + \cos. a' . \cos. b';$$

$$\text{or } \cos. (180° - C) = \sin. (180° - A) . \sin. (180° - B) . \cos. (180° - c) + \cos. (180° - A) \cos. (180° - B)$$

$$\text{or } - \cos. C = - \sin. A . \sin. B. \cos. c + \cos. A. \cos. B;$$

$$= - \sin. A. \sin. B \ (1 - \text{ver. sin. } c) + \cos. A. \cos. B;$$

and $\therefore$ $1 - \cos. C$, or, ver. sin. C =

$$1 + \cos. A . \cos. B - \sin. A. \sin. B + \text{ver. sin. } c. \sin. A. \sin. B$$

$$= 1 + \cos. (A + B) + \text{ver. sin. } c. \sin. A. \sin. B;$$

$$\therefore \text{ ver. sin. } C, \text{ or } 2 \sin.^2 \frac{C}{2} =$$

$$2 . \cos.^2 \left(\frac{A+B}{2}\right) + \text{ver. sin. } c . \sin. A \sin. B =$$

$$2 . \cos.^2 \frac{A+B}{2} \left(1 + \frac{\text{ver. sin. } c . \sin. A. \sin. B}{2 . \cos.^2 \left(\frac{A+B}{2}\right)}\right).$$

Assume $\tan.^2 \theta = \dfrac{\text{ver. sin. } c. \sin. A. \sin. B}{2 . \cos.^2 \left(\frac{A+B}{2}\right)}$, or in logarithms,

log. tan. θ =

$\frac{1}{2}$ (log. ver. sin. c + log. sin. A + log. sin. B − 2 log. cos. $\left(\frac{A+B}{2}\right)$ + 10 − log. 2).

then, ver. sin. C, or, $2 \sin.^2 \frac{C}{2} = 2 . \cos.^2 \left(\frac{A+B}{2}\right) \sec.^2 \theta$;

in logarithms, log. sin. $\frac{C}{2}$ = log. cos. $\frac{A+B}{2}$ + log. sec. θ − 10.

C, in the former Example, found independently of a, b.

$$10 - \log. 2 = 9.6989690$$
$$c = 51^\circ\ 6'\ 12''\ldots\ldots \therefore \text{ver. sin.} = 9.5706390\ *$$
$$A = 130\ 5\ 22\ldots\ldots\ldots\ldots \sin. = 9.8836842$$
$$B = 32\ 26\ 6\ldots\ldots\ldots\ldots \sin. = 9.7294422$$
$$38.8827344$$
$$\frac{A+B}{2} = 81\ 15\ 44\ldots\ldots\ldots\ldots 2\cos. = 18.3631872\ldots\ldots \cos. = 9.1815936$$
$$20.5195472$$
$$\therefore \log. \tan. \theta = 10.2597736 \ldots \sec. \theta = 10.3171290$$
$$\therefore \log. \sin. \frac{C}{2} = 9.4987226$$
$$\therefore \frac{C}{2} = 18^\circ\ 22'\ 43''.7,$$

and $C = 36^\circ\ 45'\ 27''$ nearly, as before.

If we express $1 + \cos. (A + B)$, the versed sine of the supplement of $A + B$, by suver. sin. $(A + B)$ we may employ this form for computing C

$$\text{ver. sin. } C = \text{suver. sin. } (A + B) + \text{ver. sin. } c. \sin. A. \sin. B$$
$$= \text{suver. sin. } (A + B) \sec.^2 \theta, \text{ putting}$$
$$\tan.^2 \theta = \sin. A. \sin. B. \frac{\text{ver. sin. } c}{\text{suver. sin. } (A + B)}.$$

Case V. ($a\,A\,b\,B$).

Two sides a, b, and an angle A opposite to one, are given, the angle B, and the remaining angle and side are required.

By Cor. 2, page 131, $\sin. B = \frac{\sin. A. \sin. b}{\sin. a}$,

---

* Since ver. sin. $c = 2 \sin.^2 \frac{c}{2}$, $\log.$ ver. sin. $c - \log. 2 = 2 \log. \sin. \frac{c}{2} - 10$

$$\therefore \log. \tan. \theta = \tfrac{1}{2}\left(2 \log. \sin. \frac{c}{2} + \log. \sin. A + \log. \sin. B - 2 \log. \cos. \frac{A+B}{2}\right)$$

which form is rather more convenient than the one used in the computation.

in order to find C, take the first of Naper's *Analogies*, p. 157,

then $\cot. \frac{C}{2} = \tan. \frac{1}{2}(A + B) . \frac{\cos. \frac{1}{2}(a + b)}{\cos. \frac{1}{2}(a - b)}$ .....[a]

and $\log. \cot. \frac{C}{2} =$

$\log. \tan. \frac{1}{2}(A + B) + \log. \cos. \frac{1}{2}(a + b) - \log. \cos. \frac{1}{2}(a - b)$.

C being found, c may be had from the expression $\sin. c = \sin. a . \frac{\sin. C}{\sin. A}$, or directly thus from the third of Naper's *Analogies*, p. 164,

$$\tan. \frac{c}{2} = \tan. \frac{1}{2}(a+b) \frac{\cos. \frac{1}{2}(A+B)}{\cos. \frac{1}{2}(A-B)} \text{.....[b]}$$

Example. B computed.

$A = 33° 15' 7''$	sin.	$= 9.7390354$
$b = 70\ 10\ 30$	sin.	$= 9.9734663$
		19.7125017
$a = 80\ 5\ 4$	sin.	$= 9.9934638$
$\log. \sin. B$	sin.	$= 9.7190379$

and $B = 31° 34' 37''.71$.

C computed.

$\frac{a+b}{2} = 75° 7' 47$	cos.	$= 9.4093099$
$\frac{A+B}{2} = 32\ 24\ 52$	tan.	$= 9.8027553$
		19.2120652
$\frac{a-b}{2} = 4\ 57\ 17$	cos.	$= 9.9983741$
	$\cot. \frac{C}{2}$	$= 9.2136911$

$$\therefore \frac{C}{2} = 80° 42' 38''\tfrac{7}{11},$$

and $C = 161\ 25\ 17\tfrac{3}{11}$.

c computed from C.

sin. 161° 25′ 17″ ..... = 9.5032532
sin. 30 5 4 ..... = 9.9934638
19.4967170
sin. 33 15 7 ..... = 9.7390354
log. sin. c ..... = 9.7576816

$\therefore c = 145° 5' 2''$.

c computed independently of the value of C.

$\frac{a+b}{2} = 75° 7' 47''$ ..... tan. = 10.5758962

$\frac{A+B}{2} = 32\ 24\ 52$ ..... cos. = 9.9264417
20.5023379

$\frac{A-B}{2} = 0\ 50\ 15$ ..... cos. = 9.9999536

$\therefore$ tan. $\frac{c}{2}$ = 10.5023843

$\therefore \frac{c}{2} = 72° 32' 30''\frac{1}{12}$,

and $c = 145\ 5\ 1\frac{1}{6}$.

Case VI. ($A\,a\,B\,b$).

Two angles A, B and a side (a) opposite to one of them, are given, the other side b, and, besides, the remaining side c and the angle C, are required.

b is determined from this expression, sin. b = sin. $a \frac{\text{sin. } B}{\text{sin. } A}$; C from the first of Naper's *Analogies* [1], p. 157, and c from the third [3], p. 164, as in the preceding case.

b computed.

$a = 89° 16' 53''.5$ ..... sin. $= 9.9999658$
$B = 48 \ 36 \ 0$ ..... sin. $= 9.8751256$
19.8750914
$A = 70 \ 39 \ 0$ ..... sin. $= 9.9747475$
log. sin. $b = 9.9003439$

$\therefore b = 52° 39' 4''.5.$

The sin. b = sin. $(180° - b)$, but b cannot $= 127° 20' 56''$, for since $A > B$. a must be $> b$.

c computed from the form $[b]$.

$\frac{a+b}{2} = 70° 57' 59''$ ..... tan. $= 10.4622011$
$\frac{A+B}{2} = 59 \ 37 \ 30$ ..... cos. $= 9.7038563$
20.1660574
$\frac{A-B}{2} = 11 \ 1 \ 30$ ..... cos. $= 9.9919097$
log. tan. $\frac{c}{2} = 10.1741477$

$\therefore \frac{c}{2} = 56° 11' 29''.33,$

* and $c = 112 \ 22 \ 58.6.$

---

* This last Example is taken from the *Trigonometry* of M. Legendre, who has, however, found c and C by a process different from the above. Subjoined are the data and results in French measure (F) and reduced by the Rule, page 17, to English (E).

C computed from the form [a].

$\frac{a+b}{2} = 70°\ 57'\ 59''$ ..... cos. $= 9.5133811$

$\frac{A+B}{2} = 59\ 37\ 30$ ..... tan. $= 10.2320208$

19.7454019

$\frac{a-b}{2} = 18\ 18\ 54.5$ ..... cos. $= 9.9774230$

cot. $\frac{C}{2} = 9.7679789$

$\therefore \frac{C}{2} = 59°\ 37'\ 30,$

and $C = 119\ 15\ 0.$

$a = 99°\ 20'\ 17''$ (F) $\quad B = 54°\ 0'$ (F) $\quad A = 78°.50'$ (F)

9 92 017	5 4	7 85
89 28 153	48 6	70 65
6	6	6
16.8918	36	39 0
6	$\therefore B = 48°\ 36'$ (E)	$\therefore A = 70°\ 39'$ (E)
53.508		

$\therefore a = 89°\ 16'\ 53''.5$ (E)

$c = 124°\ 86'\ 99''.3,$ $\quad b = 58°\ 50'\ 14''$ $(F),$ $\quad C = 132°\ 50'$

12 48 69.93	5 85 .014	13 25
112 38 29.37	52 65 126	119 25
6	6	6
22.97622	39.0756	15.0
6	6	$\therefore C = 119°\ 15'$ (E)
58.5732	4.536	

$\therefore c = 112°\ 22'\ 58''.6$ (E) ..... $\therefore b = 52°\ 39'\ 4''.5$ (E)

and these quantities (c, b, C,) agree with those determined in the text.

The above reductions may, more easily, be performed by the aid of the Table inserted at the end of Chap. I, p. 19.

In Case 5th, the angle C has been determined by means of B previously determined, and by the aid of one of Naper's *Analogies*; and this method, on the grounds of facility and certainty, is, perhaps, the most convenient: still, analytically considered, the determination of C does not require the previous determination of B, for by the third equation, page 147,

$$\text{cot. } A \text{. sin. } C = \text{cot. } a \text{. sin. } b - \text{cos. } C \text{. cos. } b,$$

in which A, a, b, C are alone involved. But, this form is not adapted to logarithmic computation, and in order to adapt it, we must introduce what has been called a *subsidiary* angle: thus, if we take θ such, that $\text{tan. } \theta = \text{cos. } b \text{. tan. } A$ [c]; then,

$$\frac{\text{sin. } C \text{. cos. } b}{\text{tan. } \theta} + \text{cos. } b \text{. cos. } C = \text{cot. } a \text{. sin. } b;$$

or, $\text{cos. } b \text{. (sin. } C \text{. cos. } \theta + \text{cos. } C \text{. sin. } \theta) = \text{cot. } a \text{. sin. } b \text{. sin. } \theta;$

$$\text{or, sin. } (C + \theta) = \frac{\text{cot. } a \text{. sin. } \theta \text{ sin. } b}{\text{cos. } b} = \text{tan. } b \text{. sin. } \theta \text{. cot. } a \ldots\ldots [d]$$

Hence, by deducing the logarithm of $C + \theta$, we shall know C, since θ is determined by this form,

$$\text{log. tan. } \theta = \text{log. cos. } b + \text{log. tan. } A - 10,$$

and by a similar method, that is, by the assumption of a subsidiary angle, may c be determined solely from A, a, b. It is sufficient, however, to have noted these methods, for, the computist is not recommended to avail himself of them; the preceding ones, those by which the Examples have been numerically solved, being fully adequate to the purposes of solution.

In Case 6th, C and c may be also solved by the introduction of a subsidiary angle; and its introduction, in these cases, corresponds to the Geometrical resolution of the oblique-angled into two right-angled triangles: thus, in the last Case, conceive a perpendicular (p) the arc of a great circle, to be drawn from the angle C on the base c, and let the angle contained between this perpendicular and the

side b be supposed equal to $(90^\circ - \theta)$; then, by Naper's first Rule,

$$1 \times \text{cos. } b = \text{cot. } (90^\circ - \theta) \text{ . cot. } A,$$

and tan. θ = cos. b . tan. A, which agrees with the assumption [c], p. 172.

Again, by Naper's first Rule,

$$\text{sin. } \{90^\circ - (90^\circ - \theta)\} \text{ or, sin. } \theta = \text{tan. } p \text{ . cot. } b, \text{ and}$$

$$\therefore \text{ tan. } p = \text{sin. } \theta \text{ . tan. } b.$$

And finally, by Naper's first Rule,

$$\text{cos. } \{C - (90^\circ - \theta)\} = \text{tan. } p \text{ . cot. } a, \text{ or,}$$

$$\therefore \text{ sin. } (C + \theta) = \text{sin. } \theta \text{ . tan. } b \text{ . cot. } a,$$

which agrees with the result [d] in the former page.

A great variety of instances to the preceding methods might easily be collected from *Plane Astronomy*. It is not, however, necessary to give any, since, amongst other purposes, the present Treatise is meant to be merely preparatory and subservient to the study of the latter science, and to be intelligible to the Student who may happen to be unacquainted with its technical terms and language. Astronomical Examples, stated and numerically resolved, would, indeed, be useful to the Student. One part of their utility would be, to communicate the art of translating Astronomical conditions into bare Mathematical conditions; it is not, however, the special business of a Trigonometrical Treatise to teach such art. Another part of their utility would consist in teaching the method of transforming general symbolical results and formulæ into numerical values; but, of this method sufficient specimens, it is hoped, have been given in the preceding pages.

Still, however, it is desirable to apply and illustrate the preceding formulæ; and, it happens fortunately, we can effect this without introducing either the principles or the terms of a new

science. The accounts of those *Trigonometrical Surveys*, by which the figure and dimensions of the Earth have been attempted to be determined, will furnish us with very interesting instances of exemplification.

In the next Chapter we will turn our attention to this point. We shall there perceive how results may be obtained by the direct application of the preceding methods of solution; and besides, for what reasons and by what means, those methods, in certain circumstances, are either modified, or are completely superseded by methods of approximation.

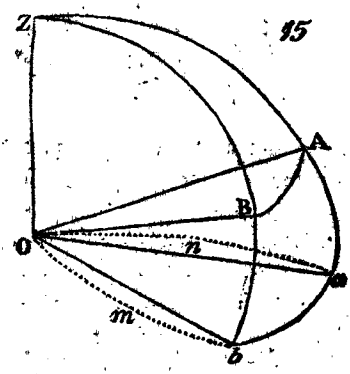
CHAP. XII.

Application of Formulæ to certain Cases that occur in the Trigonometrical Surveys.—Reduction of the Oblique to an Horizontal Angle, by the direct, by Delambre's Approximate, Method.—Excess of the Sum of the Three Angles of a Spherical Triangle above Two Right-angles.— Roy's Theorem.— Legendre's Reduction of a small Spherical to a Rectilinear Triangle.— Delambre's and Mudge's Method of Computing the Length of the Meridian.

In the Trigonometrical Survey, the two extremities of the line to be measured, are connected by a series of triangles, the angular points of which are the stations, at which, by proper instruments, the angles are observed. The sides of the triangles are determined by computation. But, since the sides cannot be determined solely from the angles, it is necessary, to measure, at least, one side of one of the series of triangles. It is usual to measure the length of a side of the first triangle, near one extremity of the line to be measured; which side is called the *Base*; and also, another line, the side of one of the triangles, near the other extremity; which latter line, serving as a test of the accuracy of the observations and computations, is called a *Base of Verification*. In the Trigonometrical Survey conducted in 1784, &c. under General Roy, the difference between the base of verification on Romney Marsh, *measured* and *computed* from the original base measured on Hounslow Heath, was found to be about 2 feet in 28,533 feet. The connecting chain of triangles by which the computation was made, extended over a space of eighty miles.

After the first important operation of the measurement of a base, and which, (the side of the first triangle,) will be a spherical arc, if the measurement be conducted by spherical triangles; or,

a straight line, if by plane triangles of the chords of the spherical arcs, the angular distances between certain objects that mark the several stations must be observed. But, as the objects, probably, are not situated on the same horizontal plane, nor on the same spherical surface, the angle observed is an oblique angle. Since, however, the lines, &c. to be computed, are supposed to lie in the surface of the earth, they are horizontal angles which are required. The oblique angle observed then must be reduced to an horizontal angle: for instance,—if the two objects be *A*, *B*; *O* the observer, and *Z* his zenith, and *a*, *b*, two points on the surface of the earth (supposed to be spherical), the observed or oblique



angle, is *AOB*, measured by *AB*, and the horizontal angle, which is required, is equal to *AZB*.

Now, by the third method of solution, Case 1, page 149,

$$\sin.^2 \frac{AZB}{2} = \frac{\sin. \frac{1}{2}(AB+ZB-ZA) \cdot \sin. \frac{1}{2}(AB+ZA-ZB)}{\sin. ZA \cdot \sin. ZB}$$

$$= \frac{\sin. \frac{1}{2}(a+H-h) \sin. \frac{1}{2}(a+h-H)}{\cos. H \cdot \cos. h}, \text{ if } AB=a,\ Aa=H,\ Bb=h$$

and in logarithms,

$$\log. \sin. \frac{AZB}{2} = \tfrac{1}{2}\{20 + \log. \sin. \tfrac{1}{2}(a+H-h) + \log. \sin. \tfrac{1}{2}$$

$$(a+h-H) - \log. \cos. H - \log. \cos. h\}$$

From this formula, a, h, H, being determined by observation, the horizontal angle may be computed. For example: in the Trigonometrical Survey, of 1787, made for the purpose of joining the Observatories of Greenwich and Paris, the angular distance of Watten and Fiennes, at Calais, was observed to be 66° 30′ 38″.9: moreover, H was $= -1''$ (a depression) and $h = 25'\ 47''.2$: hence, if O represent Calais, A and B Watten and Fiennes, $AB = 66°\ 30'\ 38''.9$.

$$\log. r^2 = 20$$

$$\tfrac{1}{2}(a + H - h) = 33°\ 2'\ 25''.35 \ldots\ldots \text{sin.} = 9.7365802$$

$$\tfrac{1}{2}(a + h - H) = 33\ 28\ 13.55 \ldots\ldots \text{sin.} = 9.7415519$$

$$39.4781321$$

$$\log. \cos. -1'' + \log. \cos. 25'\ 47''.2 \ldots\ldots = 19.9999878$$

$$\therefore \log. \sin. \frac{AZB}{2} = \tfrac{1}{2}(19.4781443)$$

$$= 9.7390721$$

$$\therefore \frac{AZB}{2} = 33°\ 15'\ 18''\tfrac{14}{32}, \text{ and } AZB = 66°\ 30'\ 36''\tfrac{7}{8}.$$

Here, the reduced or horizontal angle differs from the oblique or observed angle by a quantity a little more than 2″; and although the preceding process of reduction is not a long process, yet, for practical purposes, it is not sufficiently short, when several hundred reductions are to be made; and for this cause, M. Delambre * has investigated a formula, and constructed Tables of easy use, for finding a small correction; which, applied to the oblique angle, should give the horizontal angle. A formula similar to Delambre's, will be investigated in the Appendix.

But, whether we use the preceding direct method, or Delambre's approximate method, all observed angles, if it is proposed to conduct the operation with nicety, must be reduced to horizontal angles;

---

* See *Connoissance des Temps*, 1793; Maskelyne, *Phil. Trans.* 1797, page 451; Legendre, *Mem. Acad.* 1787, page 413; Delambre, *Mem. Inst.* 1806, p. 112; Suanberg, *Exposition des Operations faites*, &c., p. 38.

and these latter angles are spherical angles. For, if we recur to the preceding method, the angle reduced was AZB, which is equal to the inclination of the planes OZa, OZb. Hence, if from A and B, as stations, the angles subtended by BO, AO, respectively, be observed, and reduced, by the preceding method, to horizontal angles, the three stations will be projected on the surface of a sphere, and the sum of the three angles at O, a, and b, ought to exceed 180°. (Prop. 9, p. 121).

If we could determine, independently of observation, the quantity by which the sum of the three angles ought to exceed 180°, we should then be able to judge of the accuracy of observations, which it is desirable to do, since observations made on objects situated nearly in the horizon, are liable to some uncertainty. Now, the Theorem concerning the area of a spherical triangle, (Prop. 13, page 124), enables us to determine the quantity of the excess. For, the area is equal to the difference of the sum of the three angles and 180°, and consequently, since the sides of the spherical triangle described on the Earth's surface are nearly rectilinear, they may, with scarcely any error, be considered as rectilinear in the computation of the area.

This beautiful application of Albert Girard's Theorem was first made by General Roy, in the *Philosophical Transactions* for 1790, page 171, where he gave the following rule* for computing, what he terms, the spherical excess. "*From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant logarithm* 9.3267737, *and the remainder is the logarithm of the excess above* 180°, *in seconds nearly.*"

The Rule may be thus investigated,

$a+b+c-180^\circ = \frac{4x}{S}180^\circ$ (Prop. 13), but $S = 4\pi r^2$; $\therefore a+b+c - 180^\circ = \frac{180^\circ}{\pi r^2}x$; or, if a, b, c, are reduced to seconds, the

* M. Delambre finds, very expeditiously, the spherical excess by means of a chart of the triangles, a scale, and a Table of two pages. See *Mem. Inst.* 1806, pages 166, 179, 180. *Disc. Prel.*

excess in seconds $= \frac{180 \cdot 60 \cdot 60}{\pi r^2} \varkappa$; now, on the Earth's surface, 1° (taking a mean measurement) corresponds to $(60859 \cdot 1) \times 6$ feet; $\therefore$ since an arc (= radius) $= \frac{360}{2\pi}$, $1° : (60859 \cdot 1) \times 6 :: \frac{360}{2\pi} : \frac{360}{2\pi} \times (60859 \cdot 1) \times 6 =$ Earth's radius in feet.

Hence the excess in seconds

$$= \varkappa \cdot \frac{2\pi}{360} \cdot \frac{60 \cdot 60}{6^2 \times (60859 \cdot 1)^2} = \varkappa \cdot \frac{2\pi \times 10}{36 \times (60859 \cdot 1)^2};$$

$\therefore$ log. excess

$$= \log. \varkappa - \{2 (\log. 6 + \log. 60859.1) - \log. 2\pi . 10\}$$

$$= \log. \varkappa - (11.1249536 - 1.7981799) = \log. \varkappa - 9.3267737.$$

As an instance of this rule, take the Example given by General Roy, *Phil. Trans.* 1790, p. 172.

Names of the Stations.	Observed Angles.	Distances in Feet.
Hanger-Hill Tower.....(*a*)	42° 2′ 32″	(*a*) from (*b*) = 38461.12
Hampton Poor-house....(*b*)	67 55 39	(*a*) from (*c*) = 24704.7
King's Arbour.....(*c*)	70 1 48	
	179 59 59	

Hence, making the distance from (*a*) to (*c*) the base of the triangle, the perpendicular on that base will be equal to 38461.12 × sin. 42° 2′ 32″, and therefore the area of the triangle $= \frac{\text{base} \times \text{perp}^r.}{2}$

$= 24704.7 \times 19230.56 \times \sin. 42° 2' 32''.$

Computation.

log. sin. 42° 2′ 32″	= 9.8258661	8.5026328
log. 24704.7	= 4.3927761	9.3267737
log. 19230.56	= 4.2839906	−1.1758591
	18.5026328	

∴ taking away 10 for radius, 8.5026328 is the logarithm of the area in feet.

Corrs. No. = .14992,* or the spherical excess in seconds is nearly .15.

Again, as a second instance, take that given in the succeeding page;

Names of the Stations.	Observed Angles.	Distances in Feet.
Hundred Acres.....(*d*)	53° 58′ 35″.75	
Hanger Hill Tower..(*e*)	68 24 44	(*d*) from (*e*) = 71934.2
St. Ann's Hill.....(*f*)	57 36 39.5	(*d*) from (*f*) = 79211.22
	179 59 59.25	

log. sin. 53° 58′ 35″.75 = 9.9078287	9.3625207
log. 35967.1..... = 4.5559054	9.3267737
log. 79211.22..... = 4.8 87866	0.0357470 = log. 1.0858
19.3625207	or spherical excess in seconds = 1″.0858.

Third Instance, from page 428, *Phil. Trans.* 1803.

Buxter Hill.....(*g*)	76° 12′ 22″	
Dean Hill.....(*h*)	48 4 32.25	(*i*) from (*g*) = 140580.4
Dunnose.....(*i*)	55 43 • 7	(*i*) from (*h*) = 183496.2

log. sin. 55° 43′ 7″ = 9.917 1279	10.0276492
log. 70290.2..... = 4.8468947	9.3267737
log. 18349.6..... = 5.2636266	.7008755 = log. 5″.022;
20.0276492	or spherical excess = 5″.022.

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* There appears to be an error in the *Philosophical Trans.* p. 172, where the spherical excess is put down .29, which is nearly equal to 2. (14992). The cause of the error seems to be, the neglect of the divisor 2 in estimating the area of the triangle; for if we add the log. 2 to − 1.1758591, the resulting number expresses the logarithm of .29.

In this manner the spherical excess computed*, will enable the observer to examine the accuracy of his observations, and in some degree to correct them. He may then proceed to calculate the sides of the triangles by the rules of Spherical Trigonometry. But, these rules, although they must give exact results, will not give results very expeditiously. It is, however, very desirable to obtain results expeditiously, when several hundred triangles are to be solved; and, this desideratum, M. Legendre has put us in possession of, by combining sufficient exactness with conciseness in the following Theorem: "*A spherical triangle being proposed, of which the sides are very small relatively to the radius of the sphere, if from each of its angles one-third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, the sides of which are equal in length to those of the proposed spherical triangle.*" The demonstration of this rule will be given in the Appendix, but its utility may be here shewn. Suppose in a spherical triangle, two sides and the included angle to have been determined, respectively, equal to 53° 58′ 35″.75, 71934.2, 79211.22 feet; compute the spherical excess, which is equal to 1″.08 nearly (see the second Example, p. 180), then $\frac{1}{3}(1''.08) = .36$, and $53° 58' 35''.75 - .36 = 53° 58' 35''.39$: with this, as an included angle, and the given values of the two sides, resolve the triangle as a rectilinear triangle, by the method † given in page 76, Plane Trigonometry.

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* In two of the instances that have been given, the sum of three observed angles is less than 180°; ∴ in order to correct that *sum*, we must add the defect plus the spherical excess; that is, in the first instance, we must add 1″+.15; in the second, .75+1″.08. If all the observed angles were made under similar circumstances, and with equal care, we should have angles better suited to calculation than the observed angles, by adding to each of the latter, one-third of the preceding corrections 1″.15 and 1″.83. But, if two of the observations should have been made more correctly than the third, then to this third is the correction chiefly to be applied. General Roy, probably, acted thus; for his corrections, as they now stand, appear rather arbitrary.

† This method has been used, in preference to the common method, because the logarithms of the two sides were already taken out. See page 76.

log. 79211.22 ..... = 4.8987866
log. 71934.2 ..... = 4.8569354

$\therefore$ log. tan. $\theta - 10$ ..... = 0.0418512

$\therefore \theta = 47^\circ\ 45'\ 23''.1$

$\theta - 45^\circ = 2\ 45\ 23.1$

log. tan. $(\theta - 45^\circ)$ ..... = 8.6825575
cot. $26^\circ\ 59'\ 17''.69$ ..... = 10.2930544

$\therefore$ log. tan. $\left(\frac{A-B}{2}\right)$ ..... = 8.9756119

$$\therefore \frac{A-B}{2} = 5^\circ\ 24'\ 2''.325$$

$$\frac{A+B}{2} = 63\ \ 0\ 42.31$$

$$\therefore A = 68^\circ\ 24'\ 44''.635$$
$$B = 57\ \ 36\ 39.985.$$

The Side c opposite the included Angle computed.

log. 71934.2 ..... = 4.8569354
log. sin. $53^\circ\ 58'\ 35''.39$ ..... = 9.9078286
14.7647640
log. sin. $57^\circ\ 36'\ 39''.99$ ..... = 9.9265645

$\therefore$ log. c ..... = 4.8381995

$\therefore c = 68896.9.$

We must now add to A and B one-third of the spherical excess, and we shall have the spherical triangle completely resolved;* that is,

3 Angles of the Spherical Triangle.	3 Sides of the Spherical Triangle.
$C' = 53^\circ\ 58'\ 35''.75$	$c = 68896.9$
$A' = 68\ \ 24\ \ 44.995$	$a = 79211.22$
$B' = 57\ \ 36\ \ 40.345$	$b = 71934.2$
$180\ \ 0\ \ 1.09$	

* In this instance solved by this method of Legendre, the conditions of the triangle were, two sides and an included angle; but it is plain, that by the same method, a triangle with any three other conditions may be solved.

We shall have the same result as that which we have already obtained by resolving the Example according to the method employed in Case 3, page 156; but that method is rather long, and, on that account, it is superseded by the present, which, in cases like the preceding, give results sufficiently exact. It must not, however, be forgotten, that, in fact, the preceding method is a method of approximation.

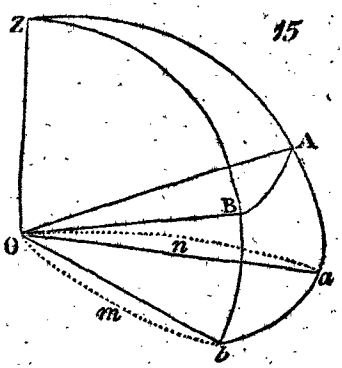
One or two cases may be brought under the conditions of the foregoing. If two sides of a spherical triangle are each nearly 180°, and the third side is very small, then, if we produce the two longer sides till they meet, a small spherical triangle will be formed, which may be resolved by the preceding method; and, from the sides and angles of this second triangle those of the first may be immediately deduced. Again, if two angles of a spherical triangle are very acute, the polar or supplemental triangle will have two sides nearly $=180°$, and the third a very small side: this case, therefore, is under the conditions of the preceding one.

Triangles solved by the foregoing methods will be spherical triangles, their sides arcs of great circles, and the computed arc of the meridian also an arc of a great circle. But, the foregoing methods have not always been employed, and they are not indispensably necessary. Delambre, in France, and Colonel Mudge, in England, do not consider the line of the meridian as a curve, but as formed of the chords of curves; and they have resolved their triangles not as spherical triangles, either by the direct, or by Legendre's approximate process, but as rectilinear triangles, the sides of which are the chords of arcs. This method then requires Theorems and formulæ different from those that have been already explained. It, however, to a certain point, goes along with the former; for it supposes the oblique angles to be reduced, either instrumentally * or by calculation, (page 176) to horizontal

* Ramsden's Theodolite, by means of an azimuth circle, gives at once the horizontal angle when the oblique angle is observed; this instrument first, we believe, shewed the spherical excess, which rarely amounts to 3″. Delambre's and Mechain's Theodolites were Borda's repeating circles of small dimensions; with these the oblique angle was observed, and then reduced by calculation (see page 176,) to the horizontal angle.

angles; and, the horizontal angles being, as it has been already explained, spherical angles, must be reduced to the angles contained by the chords of the spherical sides of the triangle. To effect this reduction, we must resolve this Problem.

Given the angle contained by the spherical arcs $a\,n\,O$, $b\,m\,O$, (Fig. 15), the angle $a\,O\,b$ contained by the chords $a\,O$, $b\,O$ is required.



Let the spherical sides $a\,b$, $b\,m\,O$, $a\,n\,O$, be represented by c, b, a;

the chords $a\,b$, $b\,O$, $a\,O$.....γ, β, α;
the spherical angle at O.....C
the angle $a\,O\,b$ of the chords.....θ

then cos. c = sin. a. sin. b . cos. C + cos. a. cos. b. (page 129).

But, cos. $c = 1 - 2\,\sin.^2 \frac{c}{2}$, and sin. $a = 2\,\sin. \frac{a}{2} \cos. \frac{a}{2}$, &c.

$$\therefore 1 - 2\,\sin.^2 \frac{c}{2} = 4\,.\,\sin. \frac{a}{2}\,.\,\cos. \frac{a}{2} \sin. b\,.\,\cos. b\,.\,\cos. C + \left\{1 - 2\,\sin.^2 \frac{a}{2}\right\}\left\{1 - 2\,\sin.^2 \frac{b}{2}\right\};$$

but, sin. $\frac{c}{2}$, sin. $\frac{a}{2}$, sin. $\frac{b}{2}$, (see p. 12,) $= \frac{\gamma}{2}, \frac{\alpha}{2}, \frac{\beta}{2}$, respectively; $\therefore$ substituting, transposing, &c.

$$2 \times \frac{\alpha^2 + \beta^2 - \gamma^2}{4} = \alpha\beta \cos.\frac{a}{2}\cos.\frac{b}{2}\cos. C + \frac{(\alpha\beta)^2}{4},$$

but (Prob. 1, pages 20, 21), $\cos.\theta = \frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}$; ∴ substituting

$$\alpha\beta \times \cos.\theta = \alpha\beta \cos.\frac{a}{2}.\cos.\frac{b}{2}\cos. C + \frac{(\alpha\beta)^2}{4},$$

and dividing by $\alpha\beta$,

$$\cos.\theta = \cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos. C + \sin.\frac{a}{2}.\sin.\frac{b}{2}.$$

From this expression, θ may be found by the method given in Case 3, page 161; thus, taking the instance given in page 177, the distance of Calais from Watten and Fiennes being in minutes and seconds of the Earth's meridian respectively 15′ 59″, 7′ 26″, we have $\frac{a}{2} = 7'\ 59''.5$, $\frac{b}{2} = 3'\ 43''$.

∴ 2 log. cos. 33° 15′ 18″.44 = 19.8446590
log. cos. 0 7 59.5 = 9.9999988
log. cos. 0 3 43 = 9.9999998
∴ 2 log. sin. M + 20 = 39.8446576
and M = 56° 44′ 41″.

Again, $90° - \frac{a+b}{4}$ = 89° 54′ 8.75;

∴ $90° - \frac{a+b}{4} + M$ = 180° − (33° 21′ 10″.25); ∴ sin. = 9.7401996

$90° - \frac{a+b}{4} - M$ = 33 9 27.75..... sin. = 9.7379440

2)19.4781436

∴ log. sin. $\frac{\theta}{2}$ = 9.7390718

and θ = 66° 30′ 36″.68.

For reasons, however, similar to those which we used when speaking of the reduction of oblique angles to horizontal angles, the direct and ordinary rules of Spherical Trigonometry are too tedious for the occasion: θ differs very little from C, and a small correction only is required. To enable the Computist to obtain

such correction and with little trouble, M. Delambre, as in the former case of reduction, page 177, has investigated a formula, and constructed Tables resembling the former Tables.

In this method, then, M. Legendre's Theorem is not employed, nor is the Theorem relative to the spherical excess requisite; for there is this criterion of the accuracy of the observations — the sum of the three angles contained by the chords deduced from the spherical angles, ought, if the observed angles were truly observed, to equal 180°. The remainder of the calculation in this method is by the common rules of Plane Trigonometry; and the arc of the meridian, if that is proposed to be determined, is composed of the sides of an irregular polygon inscribed in a circle.*

We may then, in a Trigonometrical survey, calculate the arcs and angles according to the exact rules of Spherical Trigonometry, as Boscovich has done; or, we may reduce the observed angles to angles of the chords, and calculate, by Plane Trigonometry, such reduced angles and the chords, as Delambre and Mudge have done; or we may, by a slight transformation, give to spherical triangles the properties of plane triangles, and resolve them as such, according to Legendre's method.

Delambre, indeed, indefatigable in quest of accuracy, informs us, *Mem. Inst.* page 512, 1806, that he computed by the three different methods, the whole series of triangles that extend from Dunkirk to Barcelona.

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* The whole of the process of calculation in a Trigonometrical survey has not been described. Since the Earth is not a perfect sphere, some small correction is due to its spheroidical form; the investigation of these corrections is not of great difficulty, but is omitted here, since it would be foreign to the plan and matter of the present Treatise.

CHAP. XIII.

On the Relations between the corresponding Variations of the Angles and Sides of Triangles; and, on the Means of selecting, in the Application of Trigonometrical Formulæ, the Conditions that are most favourable to accuracy of result.

THE preceding Chapter contains some illustration of the use of Trigonometrical formulæ. These formulæ are applied to certain data or conditions furnished by observation. Now, the Mathematical process is sure and infallible; but, all instrumental observation, in a greater or less degree, is liable to error. The practical result then cannot be perfectly exact: but, it will not, necessarily, be inexact to the full extent of the error of the observation. That error, according to the conditions of the case, will be variously modified by the Mathematical process. If it changes its magnitude by changing the conditions, it will be least when the conditions are of certain values. Hence, if it should happen, that we could vary the conditions, it would undoubtedly be expedient to assign them such a magnitude, that the errors of observations should least vitiate the results: that, in the words of a Mathematical statement, the error of the result should be least with a given error of observation.

These remarks stand, perhaps, in need of some illustration. The height of a tower may be determined by observing the angle which its summit makes with the horizon, and by measuring the horizontal distance between its base and the station of the observer. Now, in observing the angle a certain error may be committed: but, the error of the result (that which is Mathematically obtained) will vary as the distance between the tower and the observer is varied. If, therefore, we have it in our power, to regulate that condition, that is, if we can observe the height of the tower at what distance we please from its base, we plainly ought to select that which renders least inaccurate the result.

Again, in Astronomy, the time is determined from an observed altitude of the Sun or of a Star, from their declination and the latitude of the place. It is not then a question of mere curiosity to determine in what position, or part of the heavens, the Sun or Star ought to be observed, in order that the instrumental error supposed to be of a certain magnitude, may least vitiate the determination of the time.

The determination of the least errors is only one branch of the general Problem, which assigns, in its solution, the relations between the corresponding errors in the data and results; that is, the given errors in one or more of the conditions of the Problem, and the *consequent* errors in the results. Thus, the right ascension and declination of the Sun are computed in the Nautical Almanack from the longitude furnished by the Solar Tables and from the obliquity of the ecliptic. Now, the determination of this latter condition is subject to some error. If we assign a value to that error, we may then investigate the corresponding errors in the right ascension and declination, and, in the result of such investigation, we should necessarily include the cases, in which the original error would least affect the values of the right ascension and declination.

The *errors*, that hitherto have been spoken of, are small variations or increments in the angles and sides of rectilinear and spherical triangles. Hence, an investigation of their corresponding values will comprehend a great variety of Problems that occur in Astronomy. For instance, it would assign the effects of parallax, refraction, aberration, precession, &c. in declination, right ascension, &c. since the effects of these inequalities, always very small, may be represented by very small portions of the arcs or circles along which those effects originally take place.

It is not here intended to extend this enquiry beyond triangles; but, there are a great variety of Problems belonging to other figures and other subjects of investigation that might have been included under the class of *Errores in mixtâ Mathesi.*

This was the Title which Roger Cotes gave to his Tract* on

* Æstimatio errorum in mixta Mathesi per Variationes partium Trianguli plani et Sphærici.

this subject; and Lacaille *, in treating of the same subject, properly describes the object of Cotes's Tract to be the *determination of the limits of inevitable errors in the practice of Geometry and Astronomy.*

We purpose to treat this, as we have treated all the preceding subjects, analytically. Suppose the relation between an angle A and a side b, to be expressed by this equation,

$$\sin. A = m . \tan. b;$$

then, if A should be increased by ΔA, whilst b was increased by Δb (ΔA, Δb, representing the *entire* differences or increments of A and b), the equation belonging to the changed triangle would be

$$\sin. (A + \Delta A) = m . \tan. (b + \Delta b),$$

and the corresponding *errors* of A and b, or ΔA, Δb, would be to be determined from this equation, which is the difference of the two former, namely, from

$$\sin. (A + \Delta A) - \sin. A = m . \{\tan. (b + \Delta b) - \tan. b\}$$

If we expand † $\sin. (A + \Delta A)$, the left hand side of the equation will become

$$\frac{d . \sin. A}{dA} . \Delta A + \frac{d^2 . \sin. A}{1 . 2 . dA^2} . (\Delta A)^2 + \&c.$$

Now, in most of the cases that come under this inquiry, ΔA, whether it represents the quantity of precession, or of parallax, or of aberration, &c. is always a very small quantity: so small, that without vitiating the result, we may reject all terms involving $(\Delta A)^2$, $(\Delta A)^3$, &c.; in which case, the preceding quantity would become

$$\frac{d . \sin. A}{dA} . \Delta A.$$

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* 'Le but de l'Auteur est de determiner les limites des erreurs inevitables dans la pratique de la Geometrie et de l'Astronomie.' *Acad. des Sciences*, 1741, p. 240.

† See *Principles of Analytical Calculation*, pp. 72, 73.

In like manner, if the right hand side of the equation be evolved and the terms that involve $(\Delta b)^2$, $(\Delta b)^3$, &c. be rejected, it will be reduced to

$$\frac{d.\tan. b}{db}.\Delta b.$$

Hence ΔA, Δb, are to be determined by this equation

$$\frac{d.\sin. A}{dA}.\Delta A = m\frac{d.\tan. b}{db}.\Delta b.$$

If ΔA, Δb, should not be very small, or if considerable accuracy were required, the terms involving $(\Delta A)^2$, $(\Delta b)^2$, may be retained, in which case the equation will be

$$\frac{d.\sin. A}{dA}.\Delta A + \frac{d^2.\sin. A}{1.2.dA^2}.(\Delta A)^2 =$$

$$m\left\{\frac{d.\tan. b}{db}.\Delta b + \frac{d^2.\tan. b}{1.2.db^2}(\Delta b)^2\right\}$$

and for deducing ΔA in terms of Δb, or Δb in terms of ΔA, the solution of a quadratic would be requisite, $\frac{d.\sin. A}{dA}$, $\frac{d.\tan. b}{db}$ (see *Principles Anal. Calc.* p. 74,) are the differential coefficients of sin. A, tan. b, and are respectively equal to cos. A, sec.^{2} b.

We have taken a particular form, but if we assume a general one the method will be the same and the formula of solution similar. For instance, let X denote any function of A, and Y of b, and let the equation be

$$X = mY;$$

then the equation for determining ΔA, Δb, will be

$$\frac{dX}{dA}.\Delta A + \frac{d^2X}{1.2.dA^2}.(\Delta A)^2 + \&c. =$$

$$m\left\{\frac{dY}{db}\Delta b + \frac{d^2Y}{1.2.db^2}(\Delta b)^2, \&c.\right\}$$

and if ΔA, Δb, are very small,

$$\frac{dX}{dA}.\Delta A = m\frac{dY}{db}.\Delta b, \text{ nearly};$$

$$\text{or, } \frac{dX}{dA} \cdot \Delta A = \frac{X}{Y} \times \frac{dY}{db} \cdot \Delta b, \text{ nearly.}$$

And, in like manner, if V should be a function of C, and U of a, &c. and the finite equation of relation should be

$$X + n \cdot V + \&c. = m Y + p U + \&c.$$

n, m, &c. being constant quantities, the equation of relation between ΔA, Δa, &c. these quantities being very small, would be nearly

$$\frac{dX}{dA} \cdot \Delta A + n \cdot \frac{dV}{dC} \cdot \Delta C = m \cdot \frac{dY}{db} \Delta b + p \cdot \frac{dU}{da} \cdot \Delta a.$$

In order to facilitate the solutions of the following cases, we will prefix the values of the *differential coefficients* of sin. x, cos. x, &c.

$$*\frac{d \cdot \text{sin. } x}{dx} = \text{cos. } x, \quad \frac{d \cdot \text{cos. } x}{dx} = -\text{sin. } x, \quad \frac{d \cdot \text{tan. } x}{dx} = \text{sec.}^2 x$$

$$\frac{d \cdot \text{sec. } x}{dx} = \frac{\text{tan. } x}{\text{cos. } x}, \quad \frac{d \cdot \text{co-sec } x}{d \cdot x} = -\frac{\text{co-tan. } x}{\text{sin. } x}, \quad \frac{d \cdot \text{co-tan. } x}{dx} = -\text{co-sec.}^2 x.$$

Example 1.

In a right-angled triangle, of which one side is h, the other a, and the angle opposite h, θ, it is required to find the *error* or variation in h, from a given error in θ, (See Cotes's *Est. Errorum in mixta Mathesi*, p. 20.)

Here, $h = a \cdot \text{tan. } \theta$; $\therefore \Delta h = a \cdot \Delta \theta \cdot \frac{d \text{ tan. } \theta}{d\theta} =$

$$a \cdot \Delta \theta \cdot \text{sec.}^2 \theta = \frac{\text{sec.}^2 \theta}{\text{tan. } \theta}, h \Delta \theta$$

$$= \frac{h \cdot \Delta \theta}{\text{tan. } \theta \cdot \text{cos.}^2 \theta} = \frac{h \cdot \Delta \theta}{\text{sin. } \theta \cdot \text{cos. } \theta} = \frac{2h \cdot \Delta \theta}{\text{sin. } 2\theta};$$

consequently, if $\Delta\theta$ be given, Δh will be least when sin. 2θ is the greatest, that is, when $\theta = 45°$, and consequently, when $a = h$.

---

* $\frac{d \cdot \text{sin. } x}{dx}$, in fluxionary notation is $\frac{\dot{\text{sin. } x}}{\dot{x}}$.

Hence, if h represent the height of a tower and $\Delta\theta$, be the error of observation, (see p. 187, l. 22,) it will be most advantageous to observe the angular height of the tower at a distance about equal to its height *.

Example 2.

In a right-angled spherical triangle, where C is the right angle, and A is invariable, it is required to find the corresponding variations of the hypothenuse c and the side b.

By Naper's 1st Rule, p. 136, making the complement of A the middle part, and the radius equal 1,

$$1 \times \cos. A = \tan. b \,.\, \cot. c;$$

$\therefore$ (see p. 191, l. 12,)

$$0 = \Delta b \,.\, \sec.^2 b \,.\, \cot. c - \Delta c \,.\, \tan. b \,.\, \text{co-sec.}^2 c;$$

$$\therefore \frac{\Delta b}{\Delta c} = \frac{\tan. b}{\sec.^2 b} \times \frac{\text{co-sec.}^2 c}{\cot. c}$$

$$= \frac{\sin. b \,.\, \cos. b}{\sin. c \,.\, \cos. c} \text{ (see pp. 11, 12.)}$$

$$= \frac{\sin. 2b}{\sin. 2c} \text{ (see p. 13.)}$$

Example 3.

Let now c be invariable, and let it be required to find the ratio between the variations of the sides a, and b.

Make the complement of c the middle part, then, by Naper's second Rule, p. 136,

$$1 \times \cos. c = \cos. a \,.\, \cos. b;$$

$\therefore$ (p. 191, l. 12,)

$$0 = \Delta a \,.\, \sin. a \,.\, \cos. b - \Delta b \,.\, \sin. b \,.\, \cos. a;$$

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* Commodissum erit ad eam distantiam (AC) observationem instituere ut angulus (ACB) sit graduum 45 quamproxime.' Cotes's *Est. Errorum*, p. 20.

$$\therefore \frac{\Delta a}{\Delta b} = -\frac{\sin. b}{\cos. b} \times \frac{\cos. a}{\sin. a} = -\tan. b \times \text{co-tan. } a.$$

Example 4.

In an oblique-angled spherical triangle (SZP), if one side (PS), vary, it is required to find the corresponding variation in one of the angles (SPZ). (See Lacaille, *Mem. Acad.* 1741, p. 242).

Let the angles SPZ, SZP, ZSP, be A, C, B respectively, and a, c, b, the opposite sides; then, see p. 129, of this Work,

$$\cos. A . \sin. b . \sin. c = \cos. a - \cos. b . \cos. c;$$

$\therefore$ by p. 191,

$$-\Delta A \sin. A . \sin. b \sin. c + \Delta c . \cos. c . \cos. A . \sin. b =$$

$$\Delta c . \cos. b . \sin. c;$$

$$\therefore \Delta A = \frac{\Delta c}{\sin. A . \sin. b . \sin. c} (\cos. A . \cos. c . \sin. b - \cos. b . \sin. c)$$

$$= \Delta c \text{ (co-tan. } A \cot. c - \text{co-tan. } b \text{ co-sec. } A).$$

If Z be the zenith, P the pole, and S the Sun, then the above solution will, in the method of finding the time by equal altitudes, assign the *correction* of the time (ΔA) which is due, by reason of the *variation* or *error* (Δc) in the co-declination. (See *Astron.* p. 404).

Example 5.

In the preceding triangle (SZP) if $SZ (= a)$ vary, it is required to find the corresponding variation in the angle SPZ ($= A$). (See *Est°. Errorum*, &c. p. 21).

By p. 129, of this Work,

$$\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c};$$

$\therefore$ by p. 191,

$$-\Delta A . \sin. A = -\Delta a . \frac{\sin. a}{\sin. b . \sin. c}.$$

But, $\sin. A = \sin. C \times \frac{\sin. a}{\sin. c}$;

$$\therefore \frac{\Delta A}{\Delta a} = \frac{1}{\sin. b . \sin. C}.$$

Hence, since b is supposed to be constant, and C to be variable, ΔA is least, (Δa being given,) when $\sin. C$ is the greatest, that is, when C is a right angle.

If S, Z, &c. designate, what they were made to do, in the latter part of the preceding Example, then this solution determines the error in the time (ΔA) consequent on a given error in the observed altitude ($90° - a$), when from such altitude and the known latitude of the place, it is proposed to find the time; and, the solution also determines that the error in the time will be the least when $C (= SZP)$ is 90°, that is, when the star S is on the prime vertical. (See *Astron.* p. 408).

By similar processes we might find (as Lacaille has done, *Mem. Acad.* 1741, p. 248,) the effects produced in the right ascensions and declinations of stars, by the precession of the equinoxes. But this, and like Problems require no new or peculiar principle for their solution, the first step of which (the essential one in this class of Problems) is to be made, as in the foregoing cases, by taking the differential or fluxion of each side of the equation (see p. 191, l. 12). The other steps necessary to produce results of a certain form, must vary with the conditions of the case, and consequently cannot be anticipated and prescribed by any fixed rules.

Here terminates what specially belongs to Trigonometry. In the course of the Treatise, considerable aid has been drawn from certain auxiliary branches of science: for instance, in almost every Example, the processes and formulæ of logarithms have been introduced. Now, logarithms have neither a more intimate nor a more natural connexion with Trigonometry, than with many other branches of science. There is no eminent reason,

then, why the properties of the former should be discussed in a Treatise on the latter science. Still, since it is usual to treat together of the one and the other, the custom is here not departed from. And, accordingly, for the purpose of investigating the properties of logarithms, and for the discussion of some other subjects connected with the preceding matter, the following Appendix is now added.

APPENDIX.

In various cases of calculation it becomes necessary to multiply together numbers, to divide one number by another, and to extract the roots of numbers. We are enabled, by the common rules of Arithmetic, to perform these operations with certainty, but, not always with expedition; when the numbers are large, the process is tedious. Besides this inconvenience of tediousness, arithmetical operations requisite for a particular case cease, when the case is resolved, to be of further use: they become not subservient to the abridgment of similar operations. It was to be expected, then, that Mathematicians would soon endeavour to remove these impediments, which, in the early times of science, clogged investigation: once able to compute exactly, they naturally would wish to compute compendiously.

And, a sure, but a merely sure, method being attained, their next inquiries would be to find out a short one*. Of the invention of such a method, it belongs peculiarly to historical Treatises on Mathematics, to exhibit the first rude essays and tedious developements; but, under whatever form, or in whatever

* The Introduction to the English Translation of Briggs, states, very plainly and distinctly, the uses of logarithms, "By them all troublesome multiplications and divisions in arithmetic are avoided, and performed only by addition instead of multiplication, and by subtraction instead of division. The curious and laborious extractions of roots are also performed with great ease, as the square root of any number is found by bipartition or division by 2, &c." *Logarithmetical Arithmetic*, 1631.

terms *the method of logarithms* has been stated and explained, its principle may be reduced to this, "that all numbers are feigned to be equal to the powers of a certain assumed number." In certain simple cases, we have some numbers exactly equal to the powers of another, thus:

$4 = 2^2$ $\quad 8 = 2^3$, $\quad 16 = 2^4$, $\quad 32 = 2^5$, &c.
$9 = 3^2$, $\quad 27 = 3^3$, $\quad 81 = 3^4$, $\quad 243 = 3^5$, &c.
$100 = 10^2$, $\quad 1000 = 10^3$, $\quad 10{,}000 = 10^4$, $\quad 100{,}000 = 10^5$, &c.

What then, if all numbers are to be represented by powers of 2, ought for instance, to represent 6, which is intermediate to $4 = 2^2$, and $8 = 2^3$? If 2^x represents it, x, it is plain, must be some fractional number between the indices 2 and 3: if we take the arithmetical mean of 2 and 3, or $\frac{5}{2}$, we shall find $\frac{5}{2}$ too small to be a proper index: for $2^{\frac{5}{2}} = \sqrt{2^5} = \sqrt{32} < \sqrt{36}$; $\therefore < 6$: if we take an index greater than $\frac{5}{2}$, $\frac{8}{3}$ for instance, that will be found too large; for $2^{\frac{8}{3}} = \sqrt[3]{(256)} > \sqrt[3]{216}$; $\therefore > 6$. Hence, the index is between $\frac{5}{2}$ and $\frac{8}{3}$: and if we take a number intermediate, $\frac{31}{12}$ * for instance, we may by trials, such as have been already made, find two other *limits* or numbers between which the required index must be: and as these limits successively determined, necessarily approach to each other, in value, at each operation, the intermediate numbers must also approximate more and more to the true value of the index sought.

This is, however, nothing else than a rude method of obtaining, by trial, the index of 2: it is merely a particular way of arithmetically finding the index x in this equation $6 = 2^x$: and, together with all similar imperfect and irregular methods, it will evidently be superseded by any formula, or regular process, which from the equation

$$N = a^x,$$

exhibits x in terms of N and a.

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* $\frac{31}{12} = 2.5833$, &c. which is right as far as 2 places only, the index being 2.58496, &c.

In the instance previously adduced, a was 2, but a might have been any number, 3, or 5, or 10: its peculiar value will affect the resulting arithmetical value of x, but not the law or the general form of the expression of x, obtained from the equation $N = a^x$. The index x in the equation $N = a^x$ is called the *logarithm* of N: reversely, we may define *the logarithm of a number* N *to be that index of an assumed number* a, *that shall render true the equation*

$$N = a^x.$$

The number a is called the *base*: and it distinguishes one system of logarithms from another: with the same number the greater the base, the less the logarithm; for, if $N = a^x$, and also, $= e^y$, and a be greater than e, x must be less than y: in Briggs's System of Logarithms, $a = 10$: in Naper's, $a = 2.7182818$, &c.

But, if we admit that algebraically the equation $N = a^x$ can be solved, or that a formula can be deduced from which x may be computed, will it thence follow that the ordinary arithmetical operations will be superseded by operations of greater ease and brevity? this leads us to consider the properties of indices, such as x, in the equation $N = a^x$, or in other terms, the

Properties of Logarithms.

Let $N = a^x$, $N' = a^{x'}$, $N'' = a^{x''}$, $N''' = a^{x'''}$, &c. then $NN' = a^x \times a' = a^{x+x'}$; but by the definition, $x + x'$ is the logarithm of the number represented by NN', and x and x', by the same definition, are the logarithms of N and N'; hence we have, in other symbols,

$$\log.\ (NN') = \log.\ N + \log.\ N'.$$

If therefore we possess already computed the logarithms of numbers, instead of multiplying, when there is occasion, one number by another, it is sufficient simply to add their logarithms; and, the sum will be a logarithm corresponding to which, is the number that is the product required. For instance,

the logarithm of 2.13..... =	0.3283796
of 47.2..... =	1.6739420
	2.0023216

and, the number corresponding to 2.0023216 is 100.536, which is the product of 2.13, and 47.2; as on trial, it will be found to be.

Again, $NN'N'' = a^x \times a^{x'} \times a^{x''} = a^{x+x'+x''}$; but by the definition, $x + x' + x''$ is the logarithm of $NN'N''$, or,

$$\log. N + \log. N' + \log. N'' = \log. (NN'N'').$$

Again, $\frac{N}{N'} = \frac{a^x}{a^{x'}} = a^{x-x'}$, but by the definition, $x - x'$ is the logarithm of the number corresponding to $a^{x-x'}$, or $\frac{N}{N'}$: or,

$$\log. N - \log. N' = \log. \frac{N}{N'};$$

hence, instead of dividing, for instance, 841.32 by 5.316, subtract the logarithm of 5.316 from that of 841.32, and the remainder is a logarithm, corresponding to which is a number that is the quotient that would arise on dividing 841.32 by 5.316;

Again, $N^m = (a^x)^m = a^{mx}$, but, by the definition, mx is the logarithm of a^{mx}, or N^m; $\therefore$ in other symbols,

$$m \log. N = \log. N^m;$$

hence, instead, for instance, of multiplying 53.127, five times by itself, in order to obtain $(53.127)^6$, multiply the logarithm of 53.127 by 6, and the product is a logarithm, the number corresponding to which is the 6th power of 53.127.

Again, $N^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$: but by the definition, $\frac{x}{n}$ is the logarithm of a number represented by $a^{\frac{x}{n}}$, or $N^{\frac{1}{n}}$: or in other symbols,

$$\frac{1}{n} . \log. N = \log. N^{\frac{1}{n}}.$$

Hence, instead, for instance, of extracting the square root of 137.51 twice, in order to obtain $\sqrt[4]{(137.51)}$, divide the logarithm

of 137.51 by 4, and the result is a logarithm: and the number corresponding to it is the fourth or Biquadratic Root of 137.51.

The utility then of logarithms is manifest, in abridging arithmetical operations, or rather, in performing the complex by means of the simpler; and the inquiry is now directed towards the means and method of computing them. For this purpose, it will be necessary to solve two Problems: first to find from the equation $N = a^x$, an expression for N in terms of x and a, and then an expression for x, in terms of N and a.

Expansion of a^x.

$$a^x = \left\{1 + (a-1)\right\}^x = 1 + x.(a-1) + x.\frac{x-1}{2}.(a-1)^2$$

$$+ x.\frac{x-1}{2}.\frac{x-2}{3}.(a-1)^3 + \&c.$$ by the Binomial Theorem*:

and arranging the series by the powers of x; $a^x = 1 + x\left\{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.\right\} + qx^2 + rx^3 + \&c.$ q, r, &c. standing for certain combinations of $(a-1)^2$, $(a-1)^3$ with the numbers, 2, 3, &c.: for, it is plain, from the manner by which they must be formed, that they cannot involve the index x, or any function of it:

hence, if we put $p = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

$$a^x = 1 + px + qx^2 + rx^3 + \&c.$$

$$\text{and } a^z = 1 + pz + qz^2 + rz^3 + \&c.$$

$\therefore a^x \times a^z$, or $a^{x+z} = 1 + p(x+z) + p^2xz + q(x^2+z^2) +$ $pq(x^2z + z^2x) + \&c.$; but from the original form for a^x, substituting $x + z$ instead of x, we have

$$a^{x+z} = 1 + p(x+z) + q(x+z)^2 + r(x+z)^3 + \&c.$$

$$= 1 + p(x+z) + q.2xz + q(x^2+z^2) + r(3x^2z + 3xz^2) + \&c.$$

hence, comparing the terms that involve like powers and combinations of x and z;

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* See Woodhouse's *Principles of Analytical Calculation*, p. 35, &c.

$$p = p,$$

$$2q = p^2, \text{ and } \therefore q = \frac{p^2}{2},$$

$$3r = pq, \text{ and } \therefore r = \frac{pq}{3} = \frac{p^3}{2.3};$$

and, if s represent the coefficient of the term succeeding $r x^3$,

$$4s = rp, \text{ and } \therefore s = \frac{rp}{4} = \frac{p^4}{2.3.4};$$

$$\therefore a^x, \text{ or } N = 1 + px + \frac{p^2}{1.2}.x^2 + \frac{p^3}{1.2.3}.x^3 + \frac{p^4}{1.2.3.4}x^4 + \&c. \quad [n]$$

This series then determines the number N in terms of the base a, and the index or logarithm x; but, the reverse Problem, or that which determines x in terms of N and a is of the most consequence.

x *expressed in terms of* N *and* a.

Since $a^x = 1 + px + \frac{p^2}{1.2}x^2 + \frac{p^3}{1.2.3}x^3 + \&c.$

$$N^z = 1 + Pz + \frac{P^2}{1.2}z^2 + \frac{P^3}{1.2.3}z^3 + \&c.$$

if $P = (N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \&c.$

but, since $N = a^x$, $N^z = a^{xz} = 1 + pxz + \frac{(px)^2}{1.2}.z^2 + \&c.$

Hence, comparing the terms that are affected with like powers of z, $px = P$, $\frac{(px)^2}{1.2} = \frac{P^2}{1.2}$, the same equation in fact as $px = P$; hence $x = \frac{P}{p} = \frac{(N-1) - \frac{1}{2}.(N-1)^2 + \frac{1}{3}(N-1)^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$ [*l*]

and hence, if instead of $N = a^x$, the equation is $1 \pm N = a^x$,

we shall have $x = \frac{\pm N - \frac{1}{2}N^2 \pm \frac{1}{3}N^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$ [*L*].

This is a simple Algebraical mode of expressing x: but, it is not necessarily proper for the Arithmetical computation of x, since if N be represented by any number, 7 for instance, and a be

10, the base of the common logarithms, neither the series $6 - \frac{1}{2}(36) + \frac{1}{3}(216) - \&c.$ representing the numerator, nor the series $9 - \frac{1}{2}.(81) + \frac{1}{3}(729) - \&c.$ representing the denominator, converge: in fact, the terms of the series are larger, the more remote they are from the beginning, and consequently no number of them summed can exhibit, either exactly or nearly, the true sum. Retaining then the law of the expressions, we must now adapt them to numerical computation: and first we will shew a method of computing

$$p = (a - 1) - \tfrac{1}{2}(a - 1)^2 + \&c.$$

In the series $a^x = 1 + px + \frac{p^2}{1.2}x^2 + \frac{p^3}{1.2.3}x^3 + \&c.$

let $x = 1$, and $p = 1$, then a will have a peculiar value, and

$$be = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2.7182818284, \&c.$$

call this e, then $(e - 1) - \frac{1}{2}(e - 1)^2 + \frac{1}{3}(e - 1)^3 - \&c.$ (which is the value of p in this case) $= 1$.

In the expression for a^x put $x = \frac{1}{p}$, then

$$a^{\frac{1}{p}} = 1 + 1 + \frac{1}{1.2} + \&c. = e, \text{ and } \frac{1}{a} = e^{-p};$$

but if $e^{-p} = \frac{1}{a}$, then, by the form [*l*], page 201,

$$-p = \frac{\left(\frac{1}{a} - 1\right) - \left(\frac{1}{a} - 1\right)^2 + \frac{1}{3}\left(\frac{1}{a} - 1\right)^3 - \&c.}{(e - 1) - \frac{1}{2}(e - 1)^2 + \frac{1}{3}(e - 1)^3 - \&c.}$$

or, since the denominator $= 1$,

$$p = \frac{a - 1}{a} + \frac{1}{2}\left(\frac{a - 1}{a}\right)^2 + \frac{1}{3}\left(\frac{a - 1}{a}\right)^3 + \&c$$

Now, since $a - 1 < a$, this series always converges. Hence, in the common system, or Briggs's, in which $a = 10$,

$$p = .9 + \frac{(.9)^2}{2} + \frac{(.9)^3}{3} + \&c. = 2.3025850929, \&c.$$

Let us now endeavour to obtain the numerator by means of a converging series.

By the form [*L*], page 201, it appears, that

$$\text{if } 1 + n = a^x,\ x = \frac{1}{p}(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \&c.)$$

$$\text{and, if } 1 - n = a^y,\ y = \frac{1}{p}(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \&c.)$$

Hence,

$$\frac{1+n}{1-n} = a^{x-y}, \text{ and, } x - y = \frac{2}{p}(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \&c.)$$

$$\text{or, } \log.\left(\frac{1+n}{1-n}\right) = \frac{2}{p}(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \&c.)$$

Now, $N = \dfrac{1 + \dfrac{N-1}{N+1}}{1 - \dfrac{N-1}{N+1}}$, substitute therefore, in the preceding form, $\dfrac{N-1}{N+1}$ instead of n, and we shall have

$$\log. N = \frac{2}{p}\left\{\frac{N-1}{N+1} + \tfrac{1}{3}\left(\frac{N-1}{N+1}\right)^3 + \tfrac{1}{5}\left(\frac{N-1}{N+1}\right)^5 + \&c.\right\} \quad [\lambda]$$

and this series is plainly a converging series, for $\dfrac{N-1}{N+1}$ being a proper fraction, the terms, reckoning from the beginning of the series, are less and less.

Example.

Let it be required to compute the logarithms of 2 and 3.

$N=2$, and $\frac{N-1}{N+1}=\frac{1}{3}$

$\frac{1}{3}$ ..... $=$ .33333333

$\frac{1}{3 \cdot 3^3}$ ..... $=$ .01234567

$\frac{1}{5 \cdot 3^5}$ ..... $=$ .00082304

$\frac{1}{7 \cdot 3^7}$ ..... $=$ .00006532

$\frac{1}{9 \cdot 3^9}$ ..... $=$ .00000564

$\frac{1}{11 \cdot 3^{11}}$ ..... $=$ .00000051

$\frac{1}{13 \cdot 3^{13}}$ ..... $=$ .00000004

.34657355

$\therefore$ log. 2 $=$

$\frac{2}{2.30258509 \text{ \&c.}} \times .3465735$

$=$ .3010300 to 7 places.

$N=3$, and $\frac{N-1}{N+1}=\frac{2}{4}=\frac{1}{2}$

$\frac{1}{2}$ ..... $=$ .5

$\frac{1}{3.2^3}$ ..... $=$ .0416666666

$\frac{1}{5.2^5}$ ..... $=$ .00625

$\frac{1}{7.2^7}$ ..... $=$ .0011160714

$\frac{1}{9.2^9}$ ..... $=$ .0002170138

$\frac{1}{11.2^{11}}$ ..... $=$ .0000443892

$\frac{1}{13.2^{13}}$ ..... $=$ .0000093900

$\frac{1}{15.2^{15}}$ ..... $=$ .0000020345

$\frac{1}{17.2^{17}}$ ..... $=$ .0000004487

$\frac{1}{19.2^{19}}$ ..... $=$ .0000001003

$\frac{1}{21.2^{21}}$ ..... $=$ .0000000227

$\frac{1}{23.2^{23}}$ ..... $=$ .0000000050

.5493061422

$\therefore$ log. 3 $=$

$\frac{2}{2.30258509}$, &c. $\times$ .5493061

$=$ .4771212.

The logarithms of 2 and 3 thus computed, are the logarithms in Briggs's, or the common system, in which $a = 10$ and $(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c. = 2.30258509, \&c.$

If we take the base $= e = 2.71828182, \&c.$, in which $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \&c. = 1$, the logarithms will belong to a system called, from certain analogies, Hyperbolic, or from its inventor, Naper's; and accordingly, we shall have the hyperbolic logarithm of numbers from the preceding series, by omitting the denominator p which is equal to 1,

$$\text{since it} = (e-1) - \tfrac{1}{2}(e-1)^2 + \tfrac{1}{3}(e-1)^3 - \&c.$$

hence, hyp. log. $2 = .69314718$, and hyp. log. $3 = 1.098612, \&c.$

A constant multiplier connects one system of logarithms with another: if h be the hyperbolic logarithm of a number N, then, $h \times \frac{1}{2.30258509, \&c.}$, or $h \times .43429448, \&c.$ is the common logarithm. Generally, if the base of a system of logarithms be b, then, in that system, the logarithm of a number $N = \frac{h}{B}$,

$$\text{if } B = (b-1) - \tfrac{1}{2}(b-1)^2 + \tfrac{1}{3}(b-1)^3 - \&c.$$

or, if x be the logarithm of a number N in a system of which the base is a, then log. N (base b) $=$

$$x\frac{p}{B} = x \times \frac{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}{(b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \&c.}$$

By means of the logarithmic series, page 203, the logarithms of 2 and 3 have been computed; and, by the aid of the same series, by the properties of logarithms, and by certain simple decompositions of numbers, the logarithms of all other numbers may be found: for instance,

The logarithms of 5*, 7, 11 may be computed from the series.

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* Since $5 = \frac{10}{2}$, log. $5 =$ log. $\frac{10}{2} =$ log. $10 -$ log. $2 = 1 -$ log. $2 = 1 - .3010300 = .6989700$; $\therefore$ it is unnecessary to compute the log. 5 by the series.

The logarithms of 4, 6, 8, 9, 10, 12, 14, 15, 16, 64, &c. will follow from the properties of logarithms, for

log. $4 = \log. 2^2 = 2 \log. 2$
log. $6 = \log. (2 \times 3) = \log. 2 + \log. 3$.
log. $8 = \log. 2^3 = 3 \log. 2$.
log. $9 = \log. 3^2 = 2 \log. 3$
log. $10 = \log. (2 \times 5) = \log. 2 + \log. 5$
log. $12 = \log. 3 + \log. 4 = \log. 3 + 2 \log. 2$
log. $14 = \log. 2 + \log. 7$
log. $15 = \log. 3 + \log. 5$
log. $16 = \log. 2^4 = 4 \log. 2$
log. $64 = \log. 2^6 = 6 \log. 2$.

The logarithms of 13, 17, 19, 23, 29, &c. cannot be easily computed from the series [λ], since $\frac{N-1}{N+1}$, the larger N is, approaches to 1: with the preceding numbers 13, 17, &c. the fraction would be respectively, $\frac{12}{14}$, or $\frac{6}{7}$, $\frac{16}{18}$, or $\frac{8}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, &c. and the numbers also being prime cannot be resolved into factors; but, if they Algebraically be expressed after this manner, viz.

$$N = (N-1)\left(1 + \frac{1}{N-1}\right), \text{ then}$$

$$\log. N = \log. (N-1) + \log. \left(1 + \frac{1}{N-1}\right) =$$

$$\log. (N-1) + \frac{1}{p}\left(\frac{1}{N-1} - \frac{1}{2}\frac{1}{(N-1)^2} + \frac{1}{3}\frac{1}{(N-1)^3} - \&c.\right)$$

by [L], page 201; and thus, the logarithms may be computed by series that converge with sufficient rapidity.

For instance,

if $N = 13$; $\log. 13 = \log. 12 + \frac{1}{p}\left(\frac{1}{12} - \frac{1}{2 \cdot 12^2} + \frac{1}{3 \cdot 12^3} - \&c.\right)$

if $N = 17$; $\log. 17 = \log. 16 + \frac{1}{p}\left(\frac{1}{16} - \frac{1}{2 \cdot 16^2} + \frac{1}{3 \cdot 16^3} - \&c.\right)$

if $N = 23$; $\log. 23 = \log. 22 + \frac{1}{p}\left(\frac{1}{22} - \frac{1}{2 \cdot 22^2} + \frac{1}{3 \cdot 22^3} - \&c.\right)$

if $N = 29$; $\log. 29 = \log. 28 + \frac{1}{p}\left(\frac{1}{28} - \frac{1}{2 \cdot 28^2} + \frac{1}{3 \cdot 28^3} - \&c.\right)$

In these expressions, the logarithms of 12, 16, 22, 28, are known from the logarithms of their factors, see p. 206: and when N is a prime number, $N-1$ can be aways resolved into factors. There are, however, besides the preceding, various other artifices and methods for computing logarithms *.

But, the art of computing logarithms, and dexterity in that art, would, by themselves, be of no use in expediting calculation: if, for instance, we had to multiply 31.523 by 17.81, and to divide the product by 5.4312, it would be a most long method of performing the operation, to investigate the logarithms of these numbers. It is the circumstance of registering computed logarithms in Tables, and, by the art of printing, of multiplying such Tables, that enables us to compute quickly. The calculation of logarithms is exceedingly operose; but one man calculates for thousands, and the results of tedious operations are made subservient to the abridgment of similar ones.

By the methods already described, the logarithms of all numbers from 1 to 100000, are computed and registered in Tables. Those in common use contain the logarithms of numbers, according to Briggs's System, in which the base (a) is 10. Naper's, or the Hyperbolic logarithms are so seldom required in numerical calculation, that it is more convenient to deduce them from Briggs's, by multiplying the latter into the number 2.30258509299, &c. than to search for them in separate Tables †.

But Naper's System, in which, $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \&c. = 1$ (e = base) is, apparently, so very simple, that there must exist some substantial reason for the adoption of Briggs's. Now, in this latter system, the logarithm of 10 is 1, the logarithms of 100, or 10^2, of 1000, or 10^3, &c. are 2, 3, &c. respectively; consequently, the logarithm (L) of a number N being

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* See *Principles of Anal. Calc.* pages 142 to 183: *Phil. Trans.* 1806, p. 327: Bertrand, p. 421 to 676.

† Thomas Simpson, has given a short Table of Hyperbolic Logarithms at the end of his Fluxions: in Callet's and Hutton's Logarithms there is a Table, of a single page, for converting common into Hyperbolic Logarithms.

known, the logarithms of all numbers corresponding to $N \times 10^m$, or $\frac{N}{10^m}$ can be expressed by an alteration in L of the simplest kind.

Thus, if the logarithm of 2.7341 be .4368144, the logarithms of the numbers 27.341, 273.41, 2734.1, 27341, 273410, are 1.4368144, 2.4368144, 3.4368144, 4.4368144, 5.4368144; that is, these latter logarithms are formed from the first by merely prefixing to the decimal, 1, 2, 3, 4, 5, which are called *characteristics*, and which characteristics are always numbers one less than the number of the figures of the integers in the numbers whose logarithms are required: the reason is this,

$27.341 = 10 \times 2.7341$; $\therefore$ log. $27.341 =$ log. $10 +$ log. $2.7341 = 1.4368144$
$2734.1 = 1000 \times 2.7341$; $\therefore$ log. $2734.1 =$ log. $1000 +$ log. $2.7341 = 3.4368144$

and generally, log. $10^m \times N =$ log. $10^m +$ log. $N = m + L$

and, similarly, it is plain, that the logarithms of

$$\frac{2.7341}{10}, \frac{2.7341}{100}, \frac{2.7341}{1000}, \frac{2.7341}{10,000}, \text{ that is, of}$$

.27341, .027341, .0027341, .00027341,

must be the logarithm of 2.7341, or .4368144, subtracting, respectively, the numbers 1, 2, 3, 4, which subtraction, it is usual thus to indicate:

$\overline{1}$.4368144, $\overline{2}$.4368144, $\overline{3}$.4368144, $\overline{4}$.4368144.

The logarithm of a number (N), then, being inserted in the Tables, it is needless to insert the logarithms of those numbers that can be formed by multiplying or dividing N by 10 and powers of 10.

Hence, we are enabled to contract the size of logarithmic Tables: and this advantage is peculiarly connected with the decimal system of notation. If there had been, in common use, scales of notation, the *roots* * of which were 9, or 7, or 3: then the most *convenient* systems of logarithms would have been those,

* The *root* or *radix* of a scale is that number according to the powers of which any digit, as it is moved more and more towards the left, increases in value: in our scale the root is 10: thus $723 = 7 \times 10^2 + 2 \times 10 + 3$.

the bases of which, are 9, 7, 3, respectively. For, in such cases, after having computed the logarithm of any number N, we could immediately, by means of the characteristics, assign the logarithm of any number represented by $9^m \times N$, or $\frac{N}{9^m}$ [root = 9,] which numbers would, analogously to the present method, be denoted by merely altering the place of the point or comma that separates integers from fractions. The *root*, then, in the scale of notation ought to determine the choice of the base in a system of logarithms. We may construct logarithms with a base = 3, and then, having computed the logarithm L of a number N, the logarithms of all numbers corresponding to $3^m \times N$, and $\frac{N}{3^m}$ would be $m + L$, and $-m + L$, and, therefore, could be assigned by merely prefixing the proper characteristics; but then, in order to know the numbers corresponding to $3^m \times N$ and $\frac{N}{3^m}$, we must multiply and divide N by 3, and the powers of 3. We cannot multiply and divide by simply altering the place of the point or comma that separates integers from decimals; so that, in fact, not knowing, by inspection, such numbers as $3N$, $9N$, $27N$, and $3^m \times N$, we should be obliged to insert the logarithms of *all* numbers in the Tables.

A single instance will elucidate this statement: with a base = 3, the logarithm of 2.7341 equals .915519, then the logarithms of the numbers

	8.2023,	24.6069,	73.8207,	221.4621,
are	1.915519,	2.915519,	3.915519,	4.915519;

for, the numbers 8.2023, 24.6069, &c. are produced by multiplying 2.7341 by 3, 3^2, 3^3, 3^4, respectively; but, they are known only by actual multiplication, and consequently it would not be sufficient to insert in Tables of logarithms constructed to a base = 3, the logarithm of 2.7341 only; but, those of 8.2023, 24.6069, 73.8207, 221.4621, &c. must be also inserted: and it is plain, that the logarithms of 27.341, 273.41, 2734.1, 27341, .027341 must be also inserted. If these latter logarithms are not inserted, the Computist would be obliged to undergo the labour

of forming them, by adding to the logarithm of 2.7341, respectively, the logarithms of 10, 100, 1000, &c. computed to a base $= 3$.

This is not the sole practical inconvenience that would arise from using a system of logarithms with a base not equal to 10. We might indeed, as it has been explained, by slight Arithmetical operations, directly find the logarithms of numbers from Tables of no greater extent than those which are in use; but, the reverse operation of finding the number from the logarithm, could not at all conveniently or briefly be performed: for, the logarithm proposed might be nearly equal to a logarithm which the Tables did not contain. These considerations will, perhaps, be sufficient to shew the very great improvement that necessarily ensued on Briggs's alteration of the logarithmic base. The real value of that alteration does not seem to have been duly appreciated by writers on this subject.

For the description and use of Tables, in which the computed logarithms of numbers are recorded, the Reader is referred to the volumes of the Tables themselves: and, as nothing seems wanting to the plainness and precision of the rules therein delivered, it would be a needless accumulation of matter to insert them here. The principle however of the construction of certain small Tables for proportional parts, that are nearest the margin of every page, requires explanation. The use of these Tables is to find the logarithms of numbers consisting of more than five places. See Sherwin, p. 6, Hutton, p. 128, first edition.

Let the number composed of the first five figures or digits of the number N be n; therefore, the number next to n, or which differs from n by 1, is $n+1$; let x be the digit, which placed after the digits composing n, shall make it N, then $N = 10n + x$, and

$$\log. N = \log. (10n + x) = \log. 10n \left(1 + \frac{x}{10n}\right),$$

$$= \log. 10n + \log. \left(1 + \frac{x}{10n}\right) \quad [\text{p. } 198],$$

$$= \log. 10n + \frac{1}{p} \left\{ \frac{x}{10n} - \tfrac{1}{2} \frac{x^2}{(10n)^2} + \&c. \right\}$$

$= \log. 10n + \frac{x}{p \times 10n}$, if the terms $\frac{x^2}{2 \cdot (10n)^2}$, &c. are, on account of their minuteness, neglected.

Again, $\log. \frac{n+1}{n} = \log. \left(1 + \frac{1}{n}\right) = \frac{1}{p} \cdot \frac{1}{n}$ [neglecting $\frac{1}{2p \cdot n^2}$, &c.] consequently, $\frac{1}{pn} = \log. (n+1) - \log. n$, and

$$\frac{x}{p \times 10n} = \frac{x}{10}\Big(\log. (n+1) - \log. n\Big),$$

and from this formula the small Tables of the proportional parts may be computed: for instance,

Let $N = 678323$, then $n = 67832$, and $n + 1 = 67833$, and $\log. 67833 - \log. 67832 = 8314410 - 8314346 = 64$,

and since $x = 3$, $\frac{x}{10}\Big(\log. (n+1) - \log. n\Big) = \frac{3}{10} \times 64 = 19.2$,

(or taking the nearest whole number) $= 19$: and by putting for x, 1, 2, 3, &c. we may form the small Table which is in the page containing the number 6783, &c. thus:

	Proportional Part.	Proportional Part in the nearest integers.
$x = 1$	6.4	6
2	12.8	13
3	19.2	19
4	25.6	26
5	32	32
6	38.4	38
7	44.8	45
8	51.2	51
9	57.6	58

See Sherwin's Tables, page 6, and at number 6783, and also Hutton's, page 128.

The above proof establishes the truth of the precept for finding the logarithms of numbers consisting of more than 5 places

of figures: the other precept * which directs us to find the number corresponding to a logarithm not found exactly in Tables, may be thus proved.

Let L be the proposed logarithm, N the number: l the tabular logarithm next less; l' the tabular logarithm next greater; n, n', their corresponding numbers.

Let x be the difference of N and n, or let $N = n + x$; then,

$$\log. N = \log.(n+x) = \log. n\left(1 + \frac{x}{n}\right) = \log. n + \log.\left(1 + \frac{x}{n}\right)$$

$$\text{or, } L = l + \frac{x}{np}, \text{ nearly; } \therefore x = np\,(L - l).$$

Again,

$$\log. n' = \log.(n+1) = \log. n\left(1 + \frac{1}{n}\right) = \log. n + \log.\left(1 + \frac{1}{n}\right)$$

or $l' = l + \frac{1}{np}$; $\therefore l' - l = \frac{1}{np}$, consequently $x = \frac{L-l}{l'-l}$, and $N = n + x = n + \frac{L-l}{l'-l}$; from which expression, the precept (Sherwin, p. 8. Hutton, p. 130.) and the small Table are derived: for instance, let $L = .4414728$, then (see Tables),

$$l = .4414595, \quad n = 27635$$
$$l' = .4414752, \quad n' = 27636$$

$$\therefore L - l = 133,\ l' - l = 157, \text{ and } \frac{133}{157} = \frac{1}{10} \times \frac{1330}{157} = \frac{1}{10}$$

$$\frac{1256+74}{157} = .8 + \frac{74}{1570} = .8 + \frac{1}{100} \times \frac{7400}{1570} = .8 +$$

$$\frac{1}{100} \times \frac{6280 + 1120}{1570} = .8 + .04 + \&c. = .84 + \&c.$$ 8 and 4 being the two figures given according to the Rule and Table, 8 corresponding to 125.6, or 126 the nearest integer, and 4 to 62.8, or 63 the nearest integer.

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* Hutton, p. 130, first edition: Sherwin, p. 8, fifth edition.

Expressions for the Sine and Cosine of a Multiple Arc (see p. 40).

These expressions admit of no simple proofs; yet, we will proceed to give their proofs, principally from the necessity of the case, and, partly, on account of the importance of the formula on which they depend, and which is little known to English Students.

The object of the formula is to afford the means of computing ϕu (a function of u) when u is expressed by this equation,

$$u = x + y \,.\, fu,$$

in which $fu\,(= U)$ denotes another function of u, not necessarily the same as ϕu*.

Since $u = x + y \,.\, fu$, it is plain that u and consequently ϕu, depend on x and y, or, to use the language of Mathematicians, (which has been used on the Continent ever since the time of John Bernoulli) are *functions* of x and y; let $\phi u = V$; then, by taking the *partial* differential coefficients (see *Princ. Anal. Calc.* pp. 79, &c.) of V, we have

$$\frac{dV}{dx} = \frac{d(\phi u)}{du} \times \frac{du}{dx}, \quad \frac{dV}{dy} = \frac{d(\phi u)}{du} \times \frac{du}{dy};$$

$$\therefore \frac{dV}{dx} \times \frac{du}{dy} = \frac{dV}{dy} \times \frac{du}{dx} \ldots\ldots\ldots\ldots\ldots\ldots [a].$$

Again, take the partial differential coefficients of du, from the equation $u = x + y \,.\, U$;

then,

$$\frac{du}{dx} = 1 + y \,.\, \frac{dU}{du} \times \frac{du}{dx},$$

$$\frac{du}{dy} = U + y \,.\, \frac{dU}{du} \times \frac{du}{dy}.$$

Eliminate, by means of these 2 equations, $\frac{dU}{du}$, and

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* For instance, fu or U may in a particular case represent u^m, whilst ϕu represents sin. u, or tan. u (see *Anal. Calc.* pp. 42, &c.)

$$\frac{du}{dy} - U \cdot \frac{du}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [b].$$

Substitute in this last equation, the value of $\frac{du}{dy}$, as given in the equation $[a]$, and

$$\frac{dV}{dy} - U \cdot \frac{dV}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots [c]$$

and from these two equations, the formulæ for u and for V $(= \phi u)$ are to be derived. Thus,

$$\begin{aligned} u &= x + y \cdot fu \\ &= x + y \cdot f(x + y \cdot fu) \\ &= x + y \cdot X + y^2 \cdot A + y^3 \cdot B + \&c. \end{aligned}$$

$X = fx$, and A, B, &c. being supposed to be functions of x. Hence,

$$\frac{du}{dx} = 1 + y \cdot \frac{dX}{dx} + y^2 \cdot \frac{dA}{dx} + y^3 \cdot \frac{dB}{dx} + \&c.$$

$$\frac{du}{dy} = X + 2y \cdot A + 3y^2 \cdot B + \&c.$$

Substitute these values in the equation $[b]$, and

$$0 = X + 2y \cdot A + 3y^2 \cdot B + \&c.$$
$$- (X + y \cdot A + y^2 \cdot B) \times \left\{ 1 + y \cdot \frac{dX}{dx} + y^2 \frac{dA}{dx} + \&c. \right\}$$

Hence, comparing the coefficients of the terms involving like powers of y, we have

$$2A - A = X \frac{dX}{dx}; \text{ or, } A = \tfrac{1}{2} \frac{d(X^2)}{dx}$$

$$2B = A \cdot \frac{dX}{dx} + X \cdot \frac{dA}{dx} = \frac{d(AX)}{dx}.$$

Now $AX = X^2 \cdot \frac{dX}{dx} = \frac{1}{3} \frac{d(X^3)}{dx}$;

$$\therefore \frac{d(AX)}{dx} = \tfrac{1}{3} \frac{d\,d(X^3)}{dx^2} = \tfrac{1}{3} \frac{d^2(X^3)}{dx^2};$$

consequently, $B = \frac{1}{2 \cdot 3} \frac{d^2(X^3)}{dx^2}$, and so on.

Hence,

$$u = x + yX + \frac{y^2}{2} \cdot \frac{d(X^2)}{dx} + \frac{y^3}{2 \cdot 3} \cdot \frac{d^2(X^3)}{dx^2} + \&c.$$

and $U \left(= \frac{u - x}{y} \right) = X + \frac{y}{2} \cdot \frac{d(X^2)}{dx} + \frac{y^2}{2 \cdot 3} \cdot \frac{d(X^3)}{dx^2} + \&c.$

ϕu is to be obtained precisely in the same way. For,

$$\phi u = \phi (x + y \cdot fu) = \phi (x + yU);$$

$\therefore$ since the first term of the developement will be $\phi x (= P)$, and U has been expressed by a series of terms involving functions of x: Assume

$$V, \text{ or } \phi u = P + y \cdot Q + y^2 \cdot R + y^3 \cdot S, \&c.$$

P, Q, R, &c. being functions of x.

From this, find, as before, (p. 213, l. 19, 20,) the partial differential coefficients $\frac{dV}{dx}$, $\frac{dV}{dy}$ and substitute their values, with that of U, in the equation [c], and there results

$$0 = \cdot Q + 2Ry + 3Sy^2 + \&c.$$
$$- \left(\frac{dP}{dx} + y \cdot \frac{dQ}{dx} + y^2 \cdot \frac{dR}{dx} + \&c. \right) \times \left(X + \frac{y}{2} \frac{d(X^2)}{dx} + \&c. \right)$$

Hence, as before (p. 214, l. 19,) multiplying the two series of the latter part, and comparing the terms affected with like powers of y,

$$Q = X \frac{dP}{dx}$$

$$2R = X \frac{dQ}{dx} + \frac{dP}{2 \cdot dx} \times \frac{d(X^2)}{dx}$$
$$= X \frac{dQ}{dx} + \frac{Q}{2X} \times 2X \cdot \frac{dX}{dx}$$
$$= X \frac{dQ}{dx} + Q \frac{dX}{dx}$$
$$= \frac{d(QX)}{dx};$$

$$\therefore R = \tfrac{1}{2} \cdot \frac{d(QX)}{dx} = \tfrac{1}{2} \frac{d\left(X^2 \cdot \frac{dP}{dx}\right)}{dx}$$

and so on: Hence,

$$V, \text{ or } \phi u = P + yX \cdot \frac{dP}{dx} + \frac{y^2}{2} \frac{d\left(X^2 \cdot \frac{dP}{dx}\right)}{dx} + \&c.$$

which is the important formula which we proposed to investigate *.

We will now apply this theorem to express the cos. mA.

By p. 42, $2 \cdot \cos. mA = \alpha^m + \frac{1}{\alpha^m}$;

if, $2 \cdot \cos. A = \alpha + \frac{1}{\alpha}$.

Hence, we must, from the preceding formula, find α^m and $\frac{1}{\alpha^m}$ from this equation

$$\alpha = 2 \cdot \cos. A - \frac{1}{\alpha},$$

for this end, compare it with $u = x + y \cdot U$;

---

* See on this subject, Laplace, *Mem. Acad.* 1777, and his *Mec. Celeste*, pp. 170, &c. Lagrange, *Theorie des Fonctions Anal.* Edit. 1. pp. 101, &c. and his *Equa. Num.* Edit. 1. pp. 234, &c. and Cousin, *Astron. Physique*, pp. 15, &c.

The Student, who examines the preceding demonstration, will perceive how necessary it is, in complicated processes, to be possessed of a precise and commodious notation. In the text, the differential, in preference to the fluxionary, has been used; but, even that may, with advantage, be, in this as in like cases, superseded by Arbogast's, in which the small Roman character d is used to denote the differential coefficient. With that character, instead of $\frac{d\left(X^2 \cdot \frac{dP}{dx}\right)}{dx}$ we should write $\mathrm{D}(X^2 \cdot \mathrm{D}P)$, and so on. See *Principles of Anal. Calc.* pp. 22, 43, &c. and Preface, pp. 28, &c.

then $u = \alpha$, $x = 2 \cos. A$, $y = -1$, and

$U (= fu) = \frac{1}{\alpha}$: and the function (ϕu) of u, or of α, that is required, is α^m.

Hence, $P = \phi x = x^m = 2^m . \cos.^m A$

$$X = fx = \frac{1}{x} = \frac{1}{2 \cos. A}$$

$$\frac{dP}{dx} = m x^{m-1} = 2^{m-1} . m \cos.^{m-1} A;$$

$$\therefore X \frac{dP}{dx} = m \times 2^{m-2} . \cos.^{m-2} . A.$$

Again,

$$X^2 . \frac{dP}{dx} = m \times (2 \cos. A)^{m-3};$$

$$\therefore \frac{d\left(X^2 \frac{dP}{dx}\right)}{dx} = m . (m-3) (2 \cos. A)^{m-4}.$$

in like manner, $\frac{1}{\alpha^m} = \alpha^{-m}$ is to be obtained; in this case $\phi u = \alpha^{-m}$, and $P = x^{-m}$,

and $\frac{dP}{dx} = -m x^{-m-1}$, whence

$$X \frac{dP}{dx} = \frac{1}{2 . \cos. A} \times -m (2 \cos. A)^{-m-1} = -m . (2 \cos. A)^{-m-2}$$

$$X^2 \frac{dP}{dx} = -m . (2 \cos. A)^{-m-3},$$

and $\therefore \frac{d\left(X^2 . \frac{dP}{dx}\right)}{dx} = m . (m+3) (2 \cos. A)^{-m-4}$, &c.

* Hence, $2 . \cos. mA = \alpha^m + \frac{1}{\alpha^m} =$

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* This is only one out of the many important applications that may be made of the formula which we have demonstrated.

$$(2 \cdot \cos. A)^m - m \cdot (2 \cos. A)^{m-2} + \frac{m \cdot (m-3)}{2} (2 \cos. A)^{m-4} + \&c.$$

$$+ (2 \cos. A)^{-m} + m (2 \cos. A)^{-m-2} + \frac{m (m+3)}{2} \cdot (2 \cos. A)^{-m-4} + \&c.$$

This expression is apparently composed of two series, one arranged according to the positive, the other according to the negative, powers of 2 cos. A. But, it is to be observed, the first series is to be continued, till the terms involve, in fact, negative powers of 2 cos. A, which, in the case of m being an integer, will be all destroyed by the terms of the second series.

For instance, if $m = 1$

$$2 \cos. A = 2 \cos. A - (2 \cos. A)^{-1} - (2 \cdot \cos. A)^{-3} + \&c.$$
$$+ (2 \cos. A)^{-1} + (2 \cdot \cos. A)^{-3} + \&c.$$

Again, if $m = 2$

$$2 \cos. 2A = (2 \cos. A)^2 - 2 \cdot (2 \cos. A)^0 - (2 \cos. A)^{-2}, \&c.$$
$$+ (2 \cos. A)^{-2} + \&c.$$
$$= (2 \cos. A)^2 - 2,$$

and, similarly, it will happen, when other whole numbers are substituted for m.

Hence, in those cases in which m is represented by whole numbers, the value of 2 cos. mA may be represented by the first series alone, that series being restricted from involving any negative powers of 2 cos. A: that is, if 2 cos. $A = p$

$$2 \cdot \cos. mA = p^m - mp^{m-2} + \frac{m \cdot (m-3)}{2} p^{m-4} - \&c.$$

The series is frequently written in a reverse order; since the general term is

$$\frac{m}{m-n} \times \frac{(m-n)(m-n-1)\ldots\ldots\ldots(m-2n+1)}{1 \cdot 2\ldots\ldots\ldots\ldots n} \times p^{m-2n};$$

when $m = 2n$ the series terminates, and, the above term becomes $= 2$. If we make the series begin from this term, then

$$2 \cdot \cos. mA = \pm 2 \left\{ 1 - \frac{m^2 p^2}{1 \cdot 2 \cdot 2^2} + \frac{m^2 (m^2-4) p^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^4} - \&c. \right\}$$

the upper or lower sign taking place, as m is of the form $4s$, or $4s+2$.

If $m = 2n+1$, the preceding general term becomes mp, and then

$$2\cos. mA = \pm\left(mp - \frac{m.(m^2-1)}{1.2.3.2^2}p^3 + \frac{m(m^2-1)(m^2-9)}{1.2.3.4.5.2^4}p^5 - \&c.\right)$$

the upper or lower sign taking place as m is of the form $4s+1$, or, $4s+3$.

From these expressions for the cosine of a multiple arc, several others may be deduced. Let s and c be the sine and cosine of an arc A, then $c = \frac{p}{2}$: and

$$\cos. mA = \tfrac{1}{2}\left((2c)^m - m(2c)^{m-2} + \frac{m.(m-3)}{2}(2c)^{m-4} - \&c.\right) \quad [a]$$

or, [m odd]

$$= \pm\left(mc - \frac{m.(m^2-1)}{1.2.3}.c^3 + \frac{m(m^2-1)(m^2-9)}{1.2.3.4.5}c^5 - \&c.\right) \quad [b]$$

$$[m \text{ even}] = \pm\left(1 - \frac{m^2}{1.2}c^2 + \frac{m^2.(m^2-4)}{1.2.3.4}c^4 - \&c.\right) \ldots\ldots\ldots [c]$$

Since the expression [b] is true whatever be the arc, let the arc

instead of A, be $\frac{\pi}{2} - A$, then $\cos.\left(\frac{\pi}{2} - A\right) = \sin. A = s$, and

$\cos. m\left(\frac{\pi}{2} - A\right) = \cos.\left(\frac{m\pi}{2} - mA\right) = \pm \sin. mA$, [+ if m be of the form $4s+1$, − if of the form $4s+3$:] hence, in all cases, [m odd]

$$\sin. mA = ms - \frac{m(m^2-1)}{1.2.3}s^3 + \frac{m.(m^2-1)(m^2-9)}{1.2.3.4.5}s^5 - \&c. \quad [d]$$

Take the differential or fluxion of [a], then since [p. 98] $d(\cos. mA) = -\sin. mA.m.dA$, and $2^m.d(c^m) = 2^m \times mc^{m-1}.dc = -2^m.mc^{m-1}s.dA$, &c.

$\therefore$ dividing by $m\,dA$, we have, since s enters each term, sin. $mA =$

$$s\left((2c)^{m-1} - (m-2)(2c)^{m-3} + \frac{(m-3)(m-4)}{2}(2c)^{m-5} - \&c.\right) [e]$$

If we perform the same operations on $[b]$ and $[c]$, we shall have $[m$ odd$]$ sin. $mA =$

$$\pm s\left(1 - \frac{m^2-1}{2}c^2 + \frac{(m^2-1)(m^2-9)}{2.3.4}c^4 - \&c.\right).....[f]$$

and $[m$ even$]$ sin. $mA =$

$$\pm s\left(mc - \frac{m(m^2-4)}{2.3}c^3 + \frac{m(m^2-4)(m^2-16)}{2.3.4.5}c^5 - \&c.\right) [g]$$

If we perform a like operation on $[d]$, since $d(\text{sin. } mA) = m \cos. mA \,.\, dA$ [p. 98,] we shall have

$$\cos. mA = c\left(1 - \frac{m^2-1}{2}s^2 + \frac{(m^2-1)(m^2-9)}{2.3.4}s^4 - \&c.\right).....[h]$$

If in equation $[c]$ we put, instead of A, $\frac{\pi}{2} - A$, then

$$c = \cos.\left(\frac{\pi}{2} - A\right) = \text{sin. } A = s,$$

and $\cos. m\left(\frac{\pi}{2} - A\right) = \cos.\left(\frac{m\pi}{2} - mA\right) = \pm \cos. mA$

$[+$, if m is of form $4s$, $-$, if m is of form $4s + 2.]$

Hence, in both cases,

$$\cos. mA = 1 - \frac{m^2}{2}s^2 + \frac{m^2(m^2-4)}{2.3.4}s^4 - \&c.....[i].$$

Take the differential or fluxion of this equation $[i]$, and divide by $m\,dA$, then we have $[m$ even$]$ sin. mA

$$= c\left(ms - \frac{m(m^2-4)}{2.3}s^3 + \frac{m(m^2-4)(m^2-16)}{2.3.4.5}s^5 - \&c.\right) [k]$$

These formulæ for the sine and cosine of the multiple arc, (ten in number,) require not, as it has appeared, separate demonstrations, since the nine latter are easily derived from the first $[a]$; the attention then of the analyst ought principally to be directed to that. If in the formulæ $[a]$, $[d]$, $[e]$, $[k]$, we substitute for m, 2, 3, 4, 5, there will result, as particular instances,

the forms designated by $[c^{II}]$, $[c^{III}]$, $[c^{IV}]$, $[c^{V}]$, $[s^{III}]$, $[s^{IV}]$, &c. in pages 39, 40, 41; and if in $[f]$, $[h]$, $[i]$, we expound m by different numbers, we shall have

$$\left.\begin{array}{ll} m=3 & \text{sin. } 3A = -s(1-4c^2) \\ m=5 & \text{sin. } 5A = s(1-12c^2+16c^4) \end{array}\right\} \text{ from } [f]$$

&c.

$$\left.\begin{array}{ll} m=3 & \text{cos. } 3A = c(1-4s^2) \\ m=5 & \text{cos. } 5A = c(1-12s^2+16s^4) \end{array}\right\} \text{ from } [h]$$

&c.

$$\left.\begin{array}{ll} m=4 & \text{cos. } 4A = 1-8s^2+8s^4 \\ m=6 & \text{cos. } 6A = 1-18s^2+48s^4-32s^6 \end{array}\right\} \text{ from } [i]$$

&c.

which particular forms were not deduced in the above-mentioned pages.

Series for the Sine and Cosine.

By the form $[d]$, p. 219,

$$\text{sin. } mA = ms - \frac{m.(m^2-1)}{1.2.3}s^3 + \frac{m.(m^2-1)(m^2-9)}{2.3.4.5}s^5 - \&c.$$

Let A be very small, and m very large, and such, that $mA = x$, then $s = \text{sin. } A = \text{sin. } \frac{x}{m} = \frac{x}{m}$, nearly; and m^2-1, m^2-9, &c. $= m^2$, m^2, &c. nearly; $\therefore$ sin. mA, or,

$$\text{sin. } x = m.\frac{x}{m} - \frac{m.m^2}{2.3}.\left(\frac{x}{m}\right)^3 + \frac{m.m^2.m^2}{2.3.4.5}.\left(\frac{x}{m}\right)^5 - \&c.$$

$$= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c. \ldots\ldots\ldots\ldots\ldots [v]$$

We shall have the same result if we take the series $[k]$ in which m is even; for $c = \text{cos. } A = \text{cos. } \frac{x}{m} = 1$ nearly, and m^2-4, m^2-16, &c. $= m^2$, m^2, &c.

If in the series [*h*], or [*i*], we make the same substitutions as we have already made, we shall have

$$\text{cos. } m\,A, \text{ or cos. } x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \text{\&c.}\ldots\ldots[u]$$

Instead of computing sin. 1′ by the methods given, pp. 60 and 62, calculators, in the construction of Trigonometrical Tables, have employed the preceding series [*v*] for the sin. *x*, and under this form,

$$\text{sin. } \frac{m}{n} \cdot \frac{\pi}{2} = \frac{m}{n} \cdot \frac{\pi}{2} - \frac{m^3}{n^3} \cdot \frac{\pi^3}{2^3 . 2 . 3} + \text{\&c.}$$

putting $\frac{m}{n} \cdot \frac{\pi}{2}$ instead of *x*: they have also, availing themselves of previous computations,* taken $\pi = 3.14159\ 26535\ 89793$, and accordingly have been able to represent the above series, and the series [*u*] for the cosine, with numerical coefficients, thus,

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* Dr. Horsley, in his *Elementary Treatise on Mathematics*, p. 153, says, that this is "taking things in a preposterous order;" and, undoubtedly, it would be so in a Treatise intended specially to explain the principles of the construction of the Trigonometrical Canon, but not in a Treatise giving rules for practically constructing it with as much ease and conciseness as possible.

* $\sin. \frac{m}{n} . 90° =$	$\cos. \frac{m}{n} 90° =$
1.57079 63267 948966 $\frac{m}{n}$...[1]	1.00000 00000 000000
− 0.64596 40975 062463 $\frac{m^3}{n^3}$...[2]	− 1.23370 05501 361698 $\frac{m^2}{n^2}$
+.....7969 26262 461670 $\frac{m^5}{n^5}$...[3]	+ 0.25366 95079 010480 $\frac{m^4}{n^4}$
−.....468 17541 353187 $\frac{m^7}{n^7}$...[4]	−.....2086 34807 633530 $\frac{m^6}{n^6}$
+.....16 04411 847874 $\frac{m^9}{n^9}$	+.....91 92602 748394 $\frac{m^8}{n^8}$
−.....35988 432352 $\frac{m^{11}}{n^{11}}$	−.....2 52020 423731 $\frac{m^{10}}{n^{10}}$
+.....569 217292 $\frac{m^{13}}{n^{13}}$	+.....4710 874779 $\frac{m^{12}}{n^{12}}$
−.....6 688035 $\frac{m^{15}}{n^{15}}$	−.....63 866031 $\frac{m^{14}}{n^{14}}$
+.....60669 $\frac{m^{17}}{n^{17}}$	+.....656596 $\frac{m^{16}}{n^{16}}$
−.....438 $\frac{m^{19}}{n^{19}}$	−.....5294 $\frac{m^{18}}{n^{18}}$
	+.....34 $\frac{m^{20}}{n^{20}}$

From this series not only the sin. 1′, but the sine of any arc may be computed; for instance, let $\frac{m}{n} = \frac{1}{10}$, then computing exactly as far as seven places,

---

* Euler, *Introd. ad Anal. Inf.* p. 99. Callet's. Log. 27, 28.

since $\frac{m}{n} = .1$, [1] = .157079632

since $\frac{m^5}{n^5} = .00001$, [3] = .000000796

.157080428

since $\frac{m^3}{n^3} = .001$, [2] = .000645964

$\therefore$ sin. 9° = .156434464

The 4th term [4] and the remaining terms produce no significant figures in the 9th, 8th, &c. places.

or to 7 places, in nearest numbers, sin. 9° = .1564345. See p. 69, l. 4.

From the series for sin. $\frac{m}{n} 90^\circ$, we may, by assigning different values to $\frac{m}{n}$, deduce as many *formulæ of verification* as we please; for instance, suppose we wish to know whether sin. 20°, computed according to the methods of pages 63, 64, be rightly computed, put $\frac{m}{n} 90^\circ = 20$; $\therefore \frac{m}{n} = \frac{2}{9}$, which value is accordingly to be substituted for $\frac{m}{n}$ in the several terms of the preceding series for sin. $\frac{m}{n} \times 90^\circ$.

The sines of arcs deduced by the preceding series and the formulæ of pages 63, 64, will be expressed in parts of the radius, and be, what are called, *natural sines*; but, computation is usually conducted by means of logarithmic sines, which latter may, by the aid of the common logarithmic Tables, if the log. sines are required to seven places only, be computed by taking the logarithms of the numbers that express the natural sines; and, in order to avoid the inconvenience of negative logarithms, (for if the radius = 1, the sines are all fractions and the logarithms consequently negative) the Trigonometrical Tables are constructed to a radius = 10^{10}, the logarithm of which = 10: so that, instead of $\bar{1}.6006997$, the logarithm of .3987491, which is the natural sine of $23^\circ\ 30'$ to a radius 1, $10 + (\bar{1}.6006997)$ or 9.6006997 is made to denote the logarithm.

But, it is not absolutely necessary to compute the logarithmic from the natural sines; and indeed, if the latter consist of more than 7 places, their logarithms cannot, from the Tables in common use, be obtained: on this account it becomes necessary to shew, by what means independent of Logarithmic Tables, or requiring the aid only of the Tables that are in ordinary use, logarithmic sines may be computed to any degree of exactness.

By the form [v], $\sin. x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$

$$= x\left(1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \&c.\right)$$

Now, for the purpose of finding the composition of $1 - \frac{x^2}{2.3} + \&c.$ put $\sin. x = 0$, then (Table, p. 9. making $A = 0$) x may be 0, or π, or 2π, or 3π, &c. x may also be $-\pi$, -2π, or, -3π, &c.

hence, viewing the above series as an equation, $0, \pi, 2\pi, 3\pi$, &c. are its roots, or if we put $\frac{1}{y}$ for x, and then reduce the equation, so that the term of the highest dimensions (y^n for instance) stands first, $\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}$, &c. $-\frac{1}{\pi}, -\frac{1}{2\pi}, -\frac{1}{3\pi}$, &c. are roots of $y^n - \frac{y^{n-2}}{2.3} + \&c.$ and consequently, for reasons like those stated in p. 50, $y - \frac{1}{\pi}$, $y - \frac{1}{2\pi}$, &c. $y + \frac{1}{\pi}$, $y + \frac{1}{2\pi}$, &c. are divisors of the equation: hence,

$$y^n - \frac{y^{n-2}}{2.3} + \&c. = \left(y - \frac{1}{\pi}\right)\left(y + \frac{1}{\pi}\right)\left(y - \frac{1}{2\pi}\right)\left(y + \frac{1}{2\pi}\right) \&c.$$

$$\text{or } y^n\left(1 - \frac{1}{2.3y^2} + \&c.\right) = y\left(1 - \frac{1}{\pi y}\right) y\left(1 + \frac{1}{\pi y}\right) \&c.$$

or (dividing by y^n),

$$1 - \frac{1}{2.3y^2} + \&c. = \left(1 - \frac{1}{\pi y}\right)\left(1 + \frac{1}{\pi y}\right) \&c.$$

$$\text{and } \therefore 1-\frac{x^2}{2.3}+\&c.=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\&c.$$

$$\text{Hence, sin. } x=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\&c.$$

$$\text{or, } \left(\text{putting } x=\frac{m}{n}\cdot\frac{\pi}{2}\right)$$

$$\text{sin. } \frac{m}{n}\cdot\frac{\pi}{2}=\frac{m}{n}\cdot\frac{\pi}{2}\left(1-\frac{m^2}{4n^2}\right)\left(1-\frac{m^2}{16n^2}\right)\&c.$$

$$\text{and consequently, log. sin. } \frac{m}{n}\cdot\frac{\pi}{2}$$

$$=\log.\pi+\log.\frac{m}{2n}+\log.\left(1-\frac{m^2}{4n^2}\right)+\log.\left(1-\frac{m^2}{16n^2}\right)+\&c.$$

By a like decomposition we shall have

$$\text{cos. } \frac{m}{n}\cdot\frac{\pi}{2}=\left(1-\frac{m^2}{n^2}\right)\left(1-\frac{m^2}{9n^2}\right)\left(1-\frac{m^2}{25n^2}\right)\&c.*$$

$$\text{and log. cos. } \frac{m}{n}\cdot\frac{\pi}{2}=\log.\left(1-\frac{m^2}{n^2}\right)+\log.\left(1-\frac{m^2}{9n^2}\right)+\&c.$$

---

* If in the expression for sin. $\frac{m}{n}\cdot\frac{\pi}{2}$, we put $n-m$ instead of m, then, since sin. $\left(\frac{n-m}{n}\cdot\frac{\pi}{2}\right)=$ sin. $\left(\frac{\pi}{2}-\frac{m\pi}{n.2}\right)=$cos. $\frac{m}{n}\cdot\frac{\pi}{2}$, we have cos. $\frac{m}{n}\cdot\frac{\pi}{2}=\frac{n-m}{n}\cdot\frac{\pi}{2}\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)\left(\frac{5n-m}{4n}\right)$ &c. if we equate this expression for cos. $\frac{m}{n}\cdot\frac{\pi}{2}$ with the former one, and divide each by the common factors, there results $\frac{1}{3.3.5.5.7.7,\ \&c.}=\frac{\pi}{2}\times\frac{1}{2.2.4.4.6.6,\ \&c.}$ whence $\frac{\pi}{2}=\frac{2.2.4.4.6.6,\ \&c.}{1.3.3.5.5.7.7,\ \&c.}$ which is Wallis's expression; and many other curious results, which are not, however, the proper objects of this Treatise, might be obtained.

The preceding series may be differently expressed: since *

$$\log.\left(1-\frac{m^2}{4n^2}\right)=\log.\left(\frac{4n^2-m^2}{4n^2}\right)=\log.(2n+m)(2n-m)-2\log.2n$$

$$=\log.(2n+m)+\log.(2n-m)-2\log.2-2\log.n$$

$$\log.\sin.\frac{m}{n}.90^\circ=\log.\pi+\log.m-3\log.n-3\log.2$$

$$+\log.(2n+m)+\log.(2n-m)$$

$$+\log.\left(1-\frac{m^2}{4^2n^2}\right)+\log.\left(1-\frac{m^2}{6^2.n^2}\right)+\&c.$$

and by steps exactly similar may we obtain

$$\log.\cos.\frac{m}{n}.90^\circ=\log.(n-m)+\log.(n+m)-2\log.n$$

$$+\log.\left(1-\frac{m^2}{3^2n^2}\right)+\log.\left(1-\frac{m^2}{5^2n^2}\right)+\&c.$$

and since, (p. 163),

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* Log. $\left(1-\frac{m^2}{4n^2}\right)$, or log. $\left(1-\frac{m^2}{2^2n^2}\right)$ is not expanded, as the similar expressions log. $\left(1-\frac{m^2}{4^2n^2}\right)$, &c. are, for this reason; if expanded, it would increase the coefficient of $\frac{m^{20}}{n^{20}}$ by $\frac{1}{2^{20}}$: now $\frac{1}{2^{20}}=\frac{1}{1048576}=.00000009$, &c. or the significant figures would come in the eighth place, whereas $\frac{1}{4^{20}}=\frac{1}{1099511627776}=.00000000000009$, &c. and the significant figures do not come in till the fourteenth place: if, therefore, log. $\left(1-\frac{m^2}{2^2n^2}\right)$ had been expanded, or the powers of $\frac{1}{2^4}$ retained in the computation, we must have computed a greater number of terms, (see succeeding series, p. 229,) in order to have had the series exact to fifteen places of decimals.

$$\log.\left(1-\frac{m^2}{4^2n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{4^2n^2}+\frac{m^4}{2.4^4n^4}+\frac{m^6}{3.4^6n^6}+\&c.\right\}$$

$$\log.\left(1-\frac{m^2}{6^2n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{6^2n^2}+\frac{m^4}{2.6^4n^4}+\frac{m^6}{3.6^6n^6}+\&c.\right\}$$

$$\log.\left(1-\frac{m^2}{8^2n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{8^2n^2}+\frac{m^4}{2.8^4n^4}+\frac{m^6}{3.8^6n^6}+\&c.\right\}$$

If we sum the coefficients of $\frac{m^2}{n^2}$, $\frac{m^4}{n^4}$, &c. taking the columns vertically, that is, if we find the arithmetical value of

$$.4342944,\&c.\left\{\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\&c.\right\}\left(\text{coefficient of }\frac{m^2}{n^2}\right)$$

$$\tfrac{1}{2}.4342944,\&c.\left\{\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\&c.\right\}\left(\text{coefficient of }\frac{m^4}{n^4}\right)$$

$$\tfrac{1}{3}.4342944,\&c.\left\{\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\&c.\right\}\left(\text{coefficient of }\frac{m^6}{n^6}\right)$$

and add 10 the log. tabular radius, we shall have two Tables resembling those given in page 223.

log. sin. $\frac{m}{n}$. 90° =

log. m + log. $(2n-m)$
+ log. $(2n+m)$ − 3 log. n

[a] + 9.59405 98857 02190

[1] − 0.07002 28266 05901 $\frac{m^2}{n^2}$

[2] −.....111 72664 41661 $\frac{m^4}{n^4}$

[3] −.....3 92291 46453 $\frac{m^6}{n^6}$

−.....17292 70798 $\frac{m^8}{n^8}$

−.....843 62986 $\frac{m^{10}}{n^{10}}$

−.....43 48715 $\frac{m^{12}}{n^{12}}$

−.....2 31931 $\frac{m^{14}}{n^{14}}$

−.....12659 $\frac{m^{16}}{n^{16}}$

−.....702 $\frac{m^{18}}{n^{18}}$

−.....39 $\frac{m^{20}}{n^{20}}$

log. cos. $\frac{m}{n}$ 90° =

log. $(n-m)$ + log. $(n+m)$ − 2 log. n
+ 10

−...0.10149 48593 41892 $\frac{m^2}{n^2}$

−.....318 72940 65451 $\frac{m^4}{n^4}$

−.....20 94858 00017 $\frac{m^6}{n^6}$

−.....1 68483 48597 $\frac{m^8}{n^8}$

−.....14801 93986 $\frac{m^{10}}{n^{10}}$

−.....1365 02272 $\frac{m^{12}}{n^{12}}$

−.....129 81715 $\frac{m^{14}}{n^{14}}$

−.....12 61471 $\frac{m^{16}}{n^{16}}$

−.....1 24567 $\frac{m^{18}}{n^{18}}$

−.....12456 $\frac{m^{20}}{n^{20}}$

−.....1258 $\frac{m^{22}}{n^{22}}$

−.....128 $\frac{m^{24}}{n^{24}}$

−.....13 $\frac{m^{26}}{n^{26}}$

From these series, may logarithmic sines and cosines, independently of the values of the natural sines, be computed to 15 places; and, this inconvenience is avoided; if the natural sines had been taken, consisting of more than 7 places, no Tables in common use would give their logarithms. The logarithms indeed of the numbers m, $2n - m$, $2n + m$, &c. are supposed to be taken to 15

places, and these can be had, since the numbers will not consist of more than 6 figures; for $\frac{m}{n}$ cannot exceed $\frac{1}{2}$ *, therefore since $n = 90.60.60 = 324000$, $n + m$, $2n + m$, &c. cannot exceed 1000000.

As an instance to the preceding formula, suppose the logarithmic sine of 9° to be required: here $m = 1$, $n = 10$.

∴ log. m or log. 1	$= 0$
log. $(2n - m)$ or log. 19	$= 1.2787536009$
log. $(2n + m)$ or log. 21	$= 1.3222192947$
$[a]$	$= 9.5940598857$
	12.1950327813 $[d]$
[1]	.0007002282
[2]	1117
3 log. 10	$= 3.$
	3.0007003399 $[e]$
∴ log. sin. 9°, that is, $[d] - [e]$	$= 9.1943324414$

This is the log. sin. 9° to 10 places: and the decimal part is the logarithm of 15643446 the natural sine of 9°, found, p. 182.

Rule, with its proof, for finding the logarithmic sine and tangent of very small arcs, (see p. 89, and Introduction to Taylor's *Logarithms*, p. 17, &c.)

For the sine; to the logarithm of the arc reduced into seconds with the decimal annexed, add the constant quantity 4.6855749,

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* If $\frac{m}{n} > \frac{1}{2}$, the series for the cosine would be used for computing the sine, since sin. $(45° + A)$ = cos. $(45° - A)$: it is obvious, that the logarithms of m, $n + m$, &c. may be dispensed with entirely, by expanding log. $\left(1 - \frac{m^2}{2^2 n^2}\right)$; but then, to attain the same exactness, we must make the series consist of more terms. It is also plain, that instead of fifteen places in the numerical coefficients of the series, any number may be used. See Callet's *Logarithms*, p. 48.

and from the sum subtract one-third of the Arithmetical complement of the log. cosine; the remainder will be the logarithmic sine of the given arc.

For the tangent; to the log. arc and above constant quantity, add two-thirds of the Arithmetical complement of the log. cosine, the sum is the log. tangent of the given arc.

Proof of the Rule for the Sine.

By the series, p. 221, $\sin. x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$

$= (x$ being small$)$

$$x - \frac{x^3}{2.3} = x\left(1 - \frac{x^2}{2.3}\right) = x\left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}} \text{ nearly,}$$

hence, $\log. \sin. x = \log. x + \frac{1}{3} \log. \left(1 - \frac{x^2}{2}\right)$

$$= \log. x + \tfrac{1}{3} \log. \cos. x$$

and introducing the tabular radius 10^{10}, the logarithm of which $= 10$, $\log. \sin. x = \log. x - \frac{1}{3}(10 - \log. \cos. x)$.

Now x is in parts of the radius; let n be its value reduced to seconds, then $2r \times 3.14159$, &c. : x :: 360.60.60 : n;

consequently,

$\log. x = \log. n + \log. 2r + \log. 3.14159$, &c. $- 2 \log. 36 - \log. 1000$

$$= \log. n + 4.6855749,$$

and $\log. \sin. x = \log. n + 4.6855749 - \frac{1}{3}(10 - \log. \cos. x)$ which, since $10 - \log. \cos. x$ is the Arithmetical complement of the log. cosines, proves the rule.

Proof of the Rule for the Tangent.

$$\tan. x = \frac{\sin. x}{\cos. x} = \frac{x - \frac{x^3}{2.3}}{1 - \frac{x^2}{2}}, \text{ nearly } = \frac{x\left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}}}{1 - \frac{x^2}{2}} = \frac{x}{\left(1 - \frac{x^2}{2}\right)^{\frac{2}{3}}}$$

$$\therefore \log.\ \text{tan.}\ x = \log.\ x - \tfrac{2}{3} \log.\ \cos.\ x;$$

or $= \log.\ x + \frac{2}{3}\,(10 - \log.\ \cos.\ x)$ [introducing the tabular radius 10^{10}].

hence, as before, n being the number of seconds in the arc x,

$$\log.\ \text{tan.}\ x = \log.\ n + 4.6855749 + \tfrac{2}{3}\ \text{Arith}^{l}.\ \text{comp}^{t}.\ \log.\ \cos.\ x;$$

since $\text{cot.}\ x = \dfrac{r^2}{\text{tan.}\ x}$, $\log.$ cot. x, or $\log.$ tan. $(90° - x) =$

$$20 - (\log.\ n + 4.6855749 + \tfrac{2}{3}\ \text{Arith}^{l}.\ \text{comp}^{t}.\ \log.\ \cos.\ x).$$

These are the proofs of the two rules for finding the sine and tangent from the arc; but, there are also two rules for the reverse operation of finding the arc from the log. sine and log. tangent. These are subjoined. See Introduction to Taylor's *Logarithms*, p. 22.

Rule to find the Arc from the Sine.

To the given log. sine of a small arc add 5.3144251 and $\frac{1}{3}$ of the Arithmetical complement of log. cosine: subtract 10 from the index of the sum, and you will have the logarithm of the number of seconds with the decimal fraction of the given arc.

Rule to find the Arc from the Tangent.

To the log. tangent add 5.3144251, and from the sum subtract $\frac{2}{3}$ of the Arithmetical complement of log. cosine, and subtract 10 from the index, and you will have the logarithm of the number of seconds with the decimal fraction of the given arc.

Proof of the First Rule.

If $y =$ sine, and $x =$ arc, then $dx = \dfrac{dy}{\sqrt{(1 - y^2)}}$ .....(p. 98);

$$\therefore\ x = \int dy\,(1-y^2)^{-\frac{1}{2}} = \int dy\left(1 + \frac{y^2}{2}\right) \text{ nearly, since } y \text{ is small,}$$

$$\therefore\ x = y + \frac{y^3}{2 \cdot 3}\ (\text{nearly}) = y\left(1 + \frac{y^2}{2 \cdot 3}\right) = y\left(1 - \frac{y^2}{2}\right)^{-\frac{1}{3}}$$

$$\therefore \log. x = \log. y - \tfrac{1}{3} \log. \left(1 - \frac{y^2}{2}\right) = \log. y - \tfrac{1}{3} \log. \cos. x;$$

or to tabular radius, $\log. x = \log. y + \frac{1}{3}(10 - \log. \cos. x)$, but, p. 231, $\log. x = \log. n + 4.6855749 = \log. n + (10 - 5.3144251)$;

consequently,

$\log. n = \log. y + 5.3144251 + \frac{1}{3}$ arith^{l}. comp^{t}. $\log. \cos. x - 10$ which is the rule symbolically expressed.

Proof of the Second Rule.

$$dx = \frac{dy}{\sqrt{(1 - y^2)}}, \text{ if } t \text{ be tan. } x, \; y = \frac{t}{\sqrt{(1 + t^2)}} = t.(1 + t^2)^{-\frac{1}{2}}$$

$$= t - \tfrac{1}{2} t^3 \text{ nearly, and } y^3 = \frac{t^3}{(1 + t^2)^{\frac{3}{2}}} = t^3.(1 + t^2)^{-\frac{3}{2}}$$

$$= t^3 - \tfrac{3}{2} t^5 \text{ nearly;}$$

hence, expanding, by the Binomial Theorem, the value of dx as far as two terms, and integrating, there results

$$x = y + \frac{y^3}{2 \,.\, 3}$$

$$= t - \frac{t^3}{2} + \frac{t^3}{2 \,.\, 3} \text{ (nearly)}$$

$$= t - \frac{t^3}{3}$$

$$= t\,(1 + t^2)^{-\frac{1}{3}} \text{ (nearly).}$$

Hence, $\log. x = \log. t - \frac{1}{3} \log. \sec.^2 x = \log. t - \frac{2}{3} \log. \frac{1}{\cos. x}$, and supplying the Tabular radius, and putting for x its preceding value,

$$\log. n = \log. t + 5.3144251 - \tfrac{2}{3}(10 - \log. \cos. x) - 10$$

$= \log. t + 5.3144251 - \frac{2}{3}$ arith^{l}. comp^{t}. $\log. \cos. x - 10$, which is the rule symbolically expressed.

Examples to the Rules: with their Solutions by Sherwin's and Taylor's Tables.

Required the Arc whose Logarithmic Sine is 6.4976549.

By the Rule.

5.3144251
6.4976549
Arith^{l}. comp^{t}. = 0
11.8120800
$\therefore$ log. arc..... = 1.8120800
$\therefore$ arc..... = 64″.8754
= 1′ 4″.8754

By Sherwin.

log. sin. 2′..... = 6.7647561
log. sin. 1′..... = 6.4637261
.3010300
given log. sin..... = 6.4976549
log. sin. 1′..... = 6.4637261
339288

$$\text{arc} = 1' \frac{339288}{3010300} \times 60$$

= 1′ 6″.76.

By Taylor.

log. sin 1′ 5″..... = 6.4984882
log. sin. 1 4..... = 6.4917548
67334
given log. sin..... = 6.4976549
log. sin. 1′ 4″..... = 6.4917548
59001

$$\therefore \text{arc} = 1' 4'' + \frac{59001}{67334}$$

= 1′ 4″.876

Required the Arc whose Logarithmic Tangent is 7.1644398.

By the Rule.

5.3144251
7.1644398
12.4788649
$10 + \frac{2}{3}$ ar co. log. cos. = 10.0000003
log. arc..... = 2.4788646
$\therefore$ arc..... = 301″.2067
= 5′ 1″.2067

By Sherwin.

log. tan. 6′..... = 7.2418778
log. tan. 5..... = 7.1626964
791814
given log. tan..... = 7.1644398
log. tan..... = 7.1626964
17434

$$\therefore \text{arc} = 5' \frac{17434}{791814} \times 60''$$

= 5′ 1″.32.

By Taylor.

log. tan. 5′ . 2″ .....=7.1655821
log. tan. 5′. 1″ .....=7.1641417

14404

given log. tan.....=7.1644398
log. tan. 5′ . 1″ .....=7.1641417

2981

∴ arc.....= $5' . 1'' \frac{2981}{14404}$

= 5′ 1″.2069.

Here, if we take, in the first Example, that to be the true result which is determined by the rule, it appears that the arc is not determined from its logarithmic sine by Sherwin's Tables exactly in seconds, nor by Taylor's Tables exactly in parts of seconds (see p. 88). In the second Example, the number of seconds in the result obtained by Sherwin's Tables is right: for, although the number of seconds is not necessarily right, it may, in certain cases, happen to be right: if, instead of 7.1644398, the proposed logarithmic tangent had been 7.2022871, the arc, by Sherwin, would have been 5′ 30″, and wrong in seconds, since the true arc is 5′ 28″.633.

Amongst the independent methods of ascertaining the true result, there is none more simple than that which is called, technically, the *differential* *. If a, a', a'', &c. be successive values of a quantity, a, differing by a constant interval 1, and if the 1st, 2d, 3d, &c. differences be d', d'', d''', &c. then any intermediate value (A) distant from a by the interval x is equal to

$$a + x . d' + x . \frac{x-1}{2} . d'' + x . \frac{x-1}{2} . \frac{x-2}{3} d''' + \&c.$$

This series, by a direct application, gives the logarithmic sine and tangent of a proposed arc: thus, suppose the logarithmic tangent of 5′ 1″ 12‴ 24⁗ to be required:

a = log. tan. 5′ 0″ = 7.1626964		d'	d''
a' = log. tan. 5′ 1″ = 7.1641417		14453	− 49
a'' = log. tan. 5′ 2″ = 7.1655821		14404	− 47
a''' = log. tan. 5′ 3″ = 7.1670178		14357	

* See Wood, p. 242. Waring, 54.

Here, stopping at the 2d difference d'' and taking 48 for its mean value, and, for greater simplicity, making the series to begin from a', in which case

$$x = \frac{12 \times 60 + 24}{60.60} = \frac{62}{300}, \text{ we have}$$

$$\text{log. tan. arc required} = a' + x d' + x \cdot \frac{x-1}{2} d''$$

$$= 7.1641417 + .0014404 \times \frac{62}{300} + .0000004 = 7.1644398$$

as it ought to be. See the former Example, p. 234.

The arc cannot by a direct process be found, by means of the preceding series, from the logarithmic sine or tangent: but, thus it may be found:

Since x is small,

$$a' + x d' + x \frac{x-1}{2} d'' \,(= A) = a' + x d' - \frac{x}{2} d'' \text{ nearly.}$$

$$\text{Hence, } x = \frac{A - a'}{d' - \frac{d''}{2}}; \text{ thus, if } A = 7.1644398;$$

then (by the Tables) the logarithmic tangents next less and greater are 7.1641417, the log. tan. 5′ 1″, and, 7.1655821 the log. tan. 5′ 2″: hence, taking the differences as before,

$$x = \frac{7.1644398 - 7.1641417}{.0014404 + .0000024} = .2067,$$

and consequently the arc = 5′ 1″.2067 as before, see Example, p. 234. This method, although it does not determine the logarithmic sines and tangents of small arcs so concisely and conveniently as Dr. Maskelyne's Rule, is of very extensive use, and especially in Astronomy. It may be considered as the foundation of the common Rule given in Books of Logarithms, for finding the logarithmic sines, tangents, &c. of arcs that contain a fractional part of a minute or second: for, in mean arcs, the differences d'', d''', &c. are nothing or very small, and consequently

$$A = a + x d'.$$

Thus, to find by Sherwin's Tables the log. tan. 44° 29′ 30″.

log. tan. 44° 28′ = 9.9919143	d'	d'''
log. tan. 44° 29′ = 9.9921670	2527	0
log. tan. 44° 30′ = 9.9924197	2527	

∴ since $x = \frac{1}{2}$, log. tan. 44° 29′ 30″ = 9.9921670 + $\frac{1}{2}$ (.0002527) = 9.9922933. See p. 39. Also Sherwin, pp. 23. 24, &c. Hutton, p. 149, &c.

Demonstration of the Formula for Computing the Approximate Reduction to the Horizon. See p. 177.

Let A be the observed angle, x the correction, that is, let $A+x$ be the angle required, H and h the heights,

$$\text{then cos. } (A + x) = \frac{\text{cos. } A - \text{sin. } H. \text{ sin. } h}{\text{cos. } H . \text{ cos. } h} \text{ (p. 129).}$$

But, p. 221, Appendix,

$$\text{sin. } H = H - \frac{H^3}{2 . 3} + \text{ &c., when } H \text{ is small, } = H \text{ nearly;}$$

similarly, sin. $h = h$

$$\text{cos. } H = 1 - \frac{H^2}{2} + \text{&c.} = 1 - \frac{H^2}{2} \text{ nearly; similarly, cos. } h = 1 - \frac{h^2}{2}$$

$$\text{Hence, cos. } (A+x) = \frac{\text{cos. } A - Hh}{\left(1 - \frac{H^2}{2}\right)\left(1 - \frac{h^2}{2}\right)} = \frac{\text{cos. } A - Hh}{1 - \left(\frac{H^2}{2} + \frac{h^2}{2}\right)} =$$

$$(\text{cos. } A - Hh)\,(1 - \tfrac{1}{2}(H^2 + h^2))^{-1}$$

or, cos. A. cos. x − sin. A. sin. x = (cos. $A - Hh$) $(1 + \frac{1}{2}(H^2 + h^2))$ nearly,

but, since x is very small, cos. $x = 1$, and sin. $x = x$;

$$\therefore \text{ cos. } A - \text{sin. } A \,.\, x = \text{cos. } A - Hh + \tfrac{1}{2} \text{ cos. } A\,(H^2 + h^2)$$

$$\text{and } x = \frac{Hh - \frac{1}{2} \text{ cos. } A\,(H^2 + h^2)}{\text{sin. } A};$$

or x may be differently expressed; thus,

$$\text{make } p = \tfrac{1}{2}(H + h) \qquad \therefore p^2 - q^2 = Hh$$
$$q = \tfrac{1}{2}(H - h) \qquad p^2 + q^2 = \tfrac{1}{2}(H^2 + h^2),$$

$$\text{and } x = \frac{p^2 - q^2}{\text{sin. } A} - \frac{(p^2 + q^2) \text{ cos. } A}{\text{sin. } A}$$

$$= p^2 \cdot \frac{1-\cos. A}{\sin. A} - q^2 \cdot \frac{1+\cos. A}{\sin. A},$$

(p. 28, l. 4.) $$= p^2 \cdot \tan. \frac{A}{2} - q^2 \cdot \cot. \frac{A}{2}.$$

This is the expression given by M. Legendre, *Mem. Acad.* 1787, p. 354; also in his *Trig.* Edit. 6, p. 413. It is nearly equal to the first term of the series given by Delambre, *Connoissance des Tems*, 1793, and in the first vol. of *Measure of an Arc of the Meridian*, p. 140; by Suanberg, p. 38; and by Dr. Maskelyne, *Phil. Trans.* 1797, p. 451.

If $n = \tan. \frac{A}{2} \cdot \sin.^2 \frac{1}{2}(H+h) - \cot. \frac{A}{2} \cdot \sin.^2 \frac{1}{2}(H-h)$

then, the series deduced by the three latter Mathematicians, is

$$x = n \sec. H. \sec. h - \frac{n^2}{2} \sec.^2 H. \sec.^2 h. \cot. A. \sin. 1''$$

$$+ \tfrac{1}{2} n^3 \cdot \sec.^3 H. \sec.^3 h \left(\tfrac{1}{3} + \cot.^2 A\right) \cdot \sin.^2 1'' + \&c.$$

Here the first term, since $\sin. \frac{1}{2}(H+h)$ is nearly $= \frac{1}{2}(H+h)$, agrees with Legendre's expression.

Demonstration of Legendre's Theorem, See p. 181.

Let r be radius of sphere; a, b, c, the sides of the spherical triangle; then the sides of a similar triangle on a sphere whose radius is 1, are $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$; let also A be the spherical angle opposite to a, and A' the angle of a rectilinear triangle, the sides of which are a, b, c; then, since $\frac{1}{\sin. \frac{b}{r} \sin. \frac{c}{r}} = \left(\sin. \frac{b}{r} \sin. \frac{c}{r}\right)^{-1}$

we have, by Prop. p. 18, 149,

$$\frac{1+\cos. A}{2} = \left(\sin. \frac{S}{r} \cdot \sin. \frac{S-a}{r}\right)\left(\sin. \frac{b}{r} \cdot \sin. \frac{c}{r}\right)^{-1}$$

$$= \left(\frac{S}{r} - \frac{S^3}{2.3r^3}\right)\left(\frac{S-a}{r} - \frac{(S-a)^3}{r^3}\right) \times \left(1 - \frac{b^2}{2.3r^2} - \frac{c^2}{2.3r^2}\right)^{-1}$$

$$\times \frac{r^2}{bc}$$

$$= \frac{S(S-a)}{r^2} \times \left(1 - \frac{S^2}{2.3r^2} - \frac{(S-a)^2}{2.3r^2}\right) \times \left(1 + \frac{b^2+c^2}{2.3r^2}\right) \times \frac{r^2}{bc}$$

$$= \frac{S.(S-a)}{bc} - \frac{S.(S-a)}{bc}\left(\frac{S^2+(S-a)^2-b^2-c^2}{2.3.r^2}\right)$$

neglecting the terms that involve, in the denominator, higher powers than r^2; but,

$$S^2 + (S-a)^2 - b^2 - c^2 = \frac{(b+c+a)^2}{4} + \frac{(b+c-a)^2}{4} - b^2 - c^2$$

$$= \frac{a^2 + 2bc - b^2 - c^2}{2} = \frac{a^2 - (b-c)^2}{2}$$

$$= \frac{(a+b-c)(a-b+c)}{2} = 2.(S-c)(S-b).$$

Hence, $\frac{1+\cos. A}{2} = \frac{S.(S-a)}{bc} - \frac{2S(S-a)(S-b)(S-c)}{bc \times 6r^2}$

(Prop. 2, p. 22,) $= \frac{1+\cos. A'}{2} - \frac{bc}{2 \times 6r^2} \sin.^2 A'$;

$$\therefore \cos. A = \cos. A' - \frac{bc}{6r^2} \sin.^2 A';$$

but, $\cos. A = \cos. A' - \sin. A'.x$, if $A = A' + x$, and x be very small;

$$\therefore x = \frac{bc}{6r^2}.\ \sin. A' = \frac{1}{3r^2} \times \frac{bc.\sin. A'}{2} = \frac{\text{Area}}{3r^2},$$

and $A' = A - \frac{\text{Area}}{3r^2}$; and, in like manner, if B, C, B', C' be the other angles,

$$B' = B - \frac{\text{Area}}{3r^2}$$

$$C' = C - \frac{\text{Area}}{3r^2};$$

and $A' + B' + C' (=180°) = A + B + C - \frac{\text{Area}}{r^2}$;

or, $\frac{\text{Area}}{r^2} = A + B + C - 180°$: and thus is proved to be tru

the theorem mentioned, p. 181.

THE END.

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